Practice Exam 6 (Engineers)

Question 1

The grouped frequency table below shows the number of cars sold per week in a car showroom during a period of 200 working days.

Number of cars sold	Frequency
$26-40$	20
$41-45$	30
$46-50$	55
$51-55$	50
$56-60$	35
$61-75$	10

(i) Draw a histogram to illustrate this set of data. **[6]**

(ii) Calculate the mean number of cars sold per day. **[4]**

(iii) Construct a cumulative frequency table for the data. **[2]**

(iv) Find the lower quartile of the data. **[2]**

(v) State the modal class of the number of cars sold each week. **[1]**

Question 2

The functions f and g are defined as follows:

$$f(x) = x^2 - 1$$
$$g(x) = \cos \pi x$$

(i) Show that f has no inverse.

State why the domain $x \geq 0$ does permit an inverse of f.

In this case find an expression for the inverse of $f(x)$. **[4]**

(ii) Which of the following statements is true:

the function $g(x)$ is even,

the function $g(x)$ is odd,

the function $g(x)$ is neither even nor odd?

Give reasons for your answer. **[3]**

(iii) One of the functions f and g is periodic.

Explain which one is, and state its period. **[3]**

(iv) Show that $f(g(x)) = \frac{1}{2}(\cos 2\pi x - 1)$

Sketch its graph for the domain $1 \leq x \leq 4$. **[5]**

Question 3

(a) Differentiate $x\tan 3x$ with respect to x. **[2]**

(b) Find the stationary value (turning point) of $y=\frac{1}{x}\ln x$ for $x>0$,

and decide if this is a maximum or a minimum. **[8]**

(c) Find the equation of the tangent to the curve $y=\frac{1}{(3x-4)^3}$

at the point $(1,-1)$. **[5]**

Question 4

(a) Use the substitution $u=1+x^3$ to evaluate $\int_0^2 x^2\sqrt{1+x^3}\,\mathrm{d}x$. **[6]**

(b) Let $f(x)=\sin x$ for $0\le x\le\pi$.

(i) Sketch the curve $y=f(x)$. **[2]**

(ii) Find the area under this curve in this region. **[2]**

(iii) The area is rotated completely about the x-axis.

Find the exact value of the volume of the solid formed. **[5]**

Question 5

(a) (i) Express $\frac{x-7}{(x-2)(x^2+1)}$ in the form $\frac{A}{x-2}+\frac{B}{x^2+1}+\frac{Cx}{x^2+1}$. **[4]**

(ii) Use your result from (i) to show that

$$\int_0^1 \frac{x-7}{(x-2)(x^2+1)}\,\mathrm{d}x=\frac{3}{2}\ln 2+\frac{3\pi}{4}.$$ **[6]**

(b) Use integration by parts to find the exact value of $\int_1^2 x^2\ln x\,\mathrm{d}x$. **[5]**

Question 6

(a) (i) Show that the equation $2-x=\ln x$ has a root between 1.5 and 1.6. **[3]**

(ii) Starting with initial value $x_0=1.5$,

use the Newton-Raphson method, twice,

to give a better approximation to the root of the equation in (i)

giving your final answer correct to 4 decimal places. **[5]**

(b) Solve $5\cos 2\theta+\sin\theta=2$ for $0<\theta<2\pi$. **[7]**

Question 7

(a) Solve the differential equation

$$\frac{dy}{dx} = y^2 \sec^2 x$$

given that $y = 1$ when $x = \frac{\pi}{4}$. Express your answer in the form $y = f(x)$. **[5]**

(b) Initially a tank contains 20 litres of a solution of a dangerous chemical in water.

Pure water enters the tank at a rate of 2 litres per minute.

At the same time solution pours out of the tank at the rate of 3 litres per minute.

This reduces the concentration of chemical in the tank.

At time t minutes there are x kg of chemical in the tank.

(i) What is the concentration of chemical at time t (in kg/litre)? **[2]**

(ii) Show that x satisfies the differential equation $\frac{dx}{dt} = \frac{-3x}{20-t}$ **[2]**

(iii) Find the general solution of the differential equation. **[4]**

(iv) If initially there are 5 kg of chemical,

find how much there is after 10 minutes. **[2]**

Question 8

Let A, B and C be points with position vectors

$$\mathbf{a} = 4\mathbf{i} - 5\mathbf{j} - \mathbf{k},\ \mathbf{b} = 2\mathbf{i} - \mathbf{j} - 5\mathbf{k},\ \mathbf{c} = 4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k},$$

respectively, with respect to the origin O.

(i) Show that $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$. **[3]**

(ii) Let L_1 be the line passing through C parallel to $\overrightarrow{OA}$.

Find the vector equation of L_1. **[2]**

(iii) Show that L_1 intersects L_2,

where $L_2 = 2\mathbf{i} - 1\mathbf{j} - 5\mathbf{k} + \mu(5\mathbf{i} - 7\mathbf{j} - 2\mathbf{k})$,

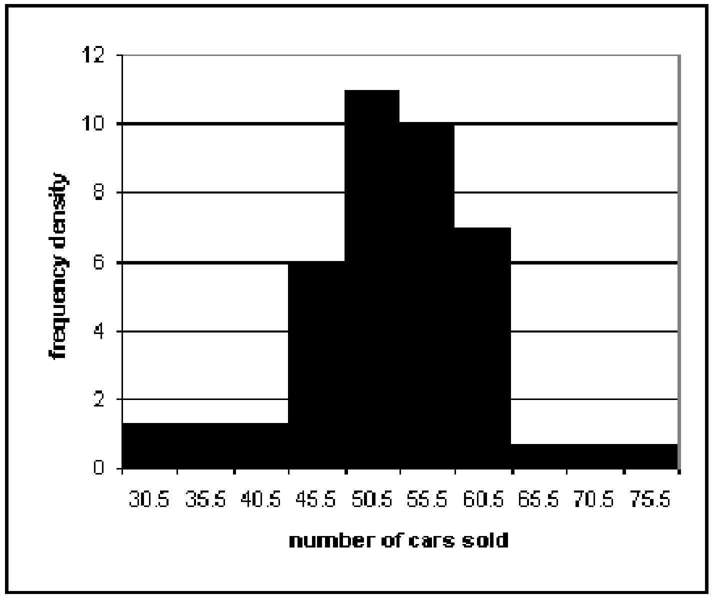
and find the point of intersection. **[6]**

(iv) Find the acute angle between the lines L_1 and L_2. **[4]**

Practice Exam 6 Answers

1

from	to	w	f	f/w
25.5	30.5	5	6.667	1.333
30.5	35.5	5	6.667	1.333
35.5	40.5	5	6.667	1.333
40.5	45.5	5	30	6
45.5	50.5	5	55	11
50.5	55.5	5	50	10
55.5	60.5	5	35	7
60.5	65.5	5	3.333	0.667
65.5	70.5	5	3.333	0.667
70.5	75.5	5	3.333	0.667



Labels on horizontal axis.	**[1]**
Horizontal scale (accept 40/40.5/41 on boundary).	**[1]**
Labels on vertical axis.	**[1]**
Vertical scale.	**[1]**
Bars right height and width.	**[2]**

(ii)

		x	f	xf
25.5	40.5	33	20	660
40.5	45.5	43	30	1290
45.5	50.5	48	55	2640
50.5	55.5	53	50	2650
55.5	60.5	58	35	2030
60.5	75.5	68	10	680
			200	9950

x column **[1]**

xf column **[2]**

mean =

$$\frac{\sum xf}{\sum f} = \frac{9950}{200} = 49.75\,.$$ **[1]**

lcb	ucb	f	cf
			0
25.5	40.5	20	20
40.5	45.5	30	50
45.5	50.5	55	105
50.5	55.5	50	155
55.5	60.5	35	190
60.5	75.5	10	200

(iii) See above **[2]**

(iv) lower quartile = 45.5 (accept 45 or 46) **[2]**

(v) modal class is 46 – 50 **[1]**

2(i) $f(1)=0,\ f(-1)=0$, so the function is not one-one. **[1]**

If $x_1 \geq 0$ and $x_2 \geq 0$ and $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$

(or other suitable explanation). **[1]**

$f^{-1}(x)=\sqrt{x+1} \qquad x \geq -1$ **[2]**

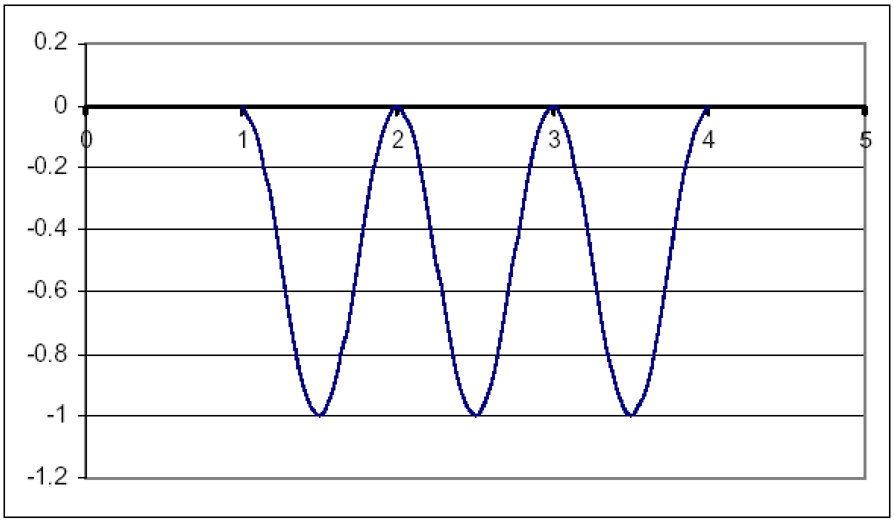
(ii) Since $\cos(\pi x)=\cos(-\pi x)$, **[2]**

the function $g(x)$ is even. **[1]**

(iii) Since $g(x+2)=\cos(\pi(x+2))=\cos(\pi x+2\pi)=\cos(\pi x)=g(x)$ **[1]**

the function $g(x)$ is periodic with period 2. **[2]**

(iv) $f(g(x))=(\cos(\pi x))^2-1=\cos^2(\pi x)-1=-\sin^2(\pi x)=\frac{1}{2}(\cos 2\pi x-1)$ **[2]**



shape **[1]**

position on x-axis **[1]**

position on y-axis **[1]**

3(a) $\frac{dy}{dx}=\tan 3x+x\sec^2 3x\times 3=\tan 3x+3x\sec^2 3x$ **[2]**

(b) $\frac{dy}{dx}=-\frac{1}{x^2}\ln x+\frac{1}{x}\frac{1}{x}=\frac{1}{x^2}(1-\ln x)$ **[2]**

$=0$ at a stationary point. **[1]**

Then $\ln x=1,\ x=e(\approx 2.72)$. **[1]**

Here $y=\frac{1}{e}\ln(e)=\frac{1}{e}(\approx 0.368)$. **[1]**

$\frac{d^2y}{dx^2}=-\frac{2}{x^3}(1-\ln x)+\frac{1}{x^2}\left(0-\frac{1}{x}\right)=-\frac{1}{x^3}(2-2\ln x+1)=-\frac{3-2\ln x}{x^3}$.

When $x=e$, $\frac{d^2y}{dx^2}=-\frac{3-2\ln(e)}{e^3}=-\frac{1}{e^3}<0$ **[2]**

So the stationary value is a maximum. **[1]**

(c) $\frac{dy}{dx}=\frac{-3}{(3x-4)^4}3=\frac{-9}{(3x-4)^4}$ **[1]**

$=-9$ at the point (1,–1). **[1]**

The equation of the tangent is $y=-9x+c$. **[1]**

$-1=-9\times 1+c,\ c=8$. **[1]**

$y=-9x+8$ **[1]**

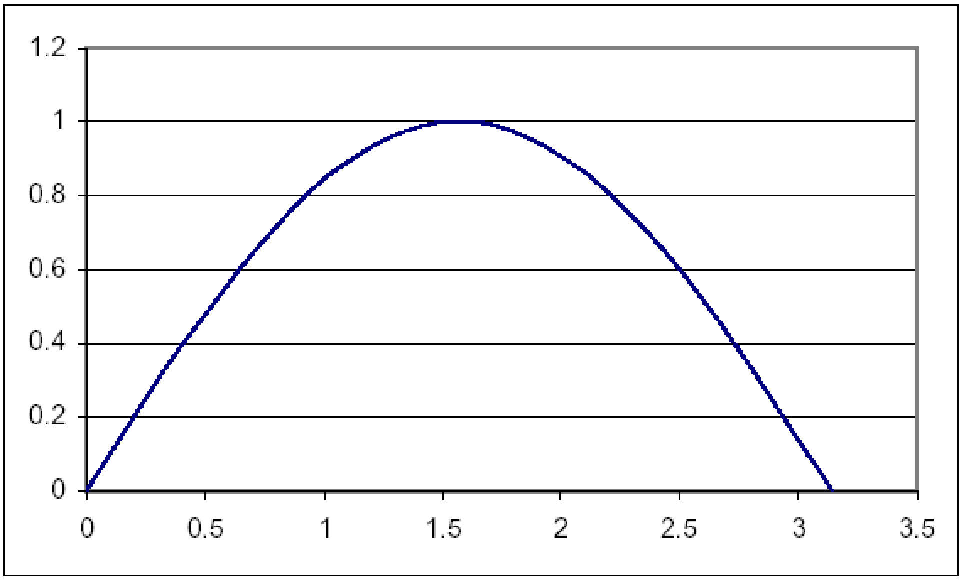
4(a) $u = 1 + x^3,\ \mathrm{d}u = 3x^2\mathrm{d}x.$ **[2]**

$$\int_0^2 x^2\sqrt{1+x^3}\,\mathrm{d}x = \int_1^9 \frac{1}{3}\sqrt{u}\,\mathrm{d}u$$ **[2]**

$$= \left[\frac{2}{9}u^{3/2}\right]_1^9$$ **[1]**

$$= \frac{2}{9}[27-1] = \frac{52}{9}(\approx 5.78)$$ **[1]**

(b)(i)



shape **[1]**

limits **[1]**

(ii) $$\int_0^\pi \sin x\,\mathrm{d}x = [-\cos x]_0^\pi$$ **[1]**

$$= [1+1] = 2$$ **[1]**

(iii) $$\pi\int_0^\pi (\sin x)^2\,\mathrm{d}x = \pi\int_0^\pi \left(\frac{1}{2}(1-\cos 2x)\right)\mathrm{d}x$$ **[2]**

$$= \pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^\pi$$ **[1]**

$$= \pi\left[\frac{\pi}{2} - 0 - 0 + 0\right]$$ **[1]**

$$= \frac{\pi^2}{2}(\approx 4.93).$$ **[1]**

5(a)(i) $\frac{A}{x-2}+\frac{B}{x^2+1}+\frac{Cx}{x^2+1}=\frac{A(x^2+1)+B(x-2)+Cx(x-2)}{(x-2)(x^2+1)}$. **[1]**

This equals $\frac{x-7}{(x-2)(x^2+1)}$

$x=2 \Rightarrow 5A=-5 \Rightarrow A=-1$. **[1]**

$x=0 \Rightarrow A-2B=-7 \Rightarrow B=3$. **[1]**

Equating coefficients of $x^2 \Rightarrow A+C=0 \Rightarrow C=1$. **[1]**

Given expression equals $\frac{-1}{x-2}+\frac{3}{x^2+1}+\frac{x}{x^2+1}$.

(ii) Given integral equals

$\int_0^1\left(\frac{-1}{x-2}+\frac{3}{x^2+1}+\frac{x}{x^2+1}\right)dx=\left[-\ln|x-2|+3\tan^{-1}x+\frac{1}{2}\ln(x^2+1)\right]_0^1$ **[3]**

$=\left[-\ln(1)+3\tan^{-1}1+\frac{1}{2}\ln(2)+\ln(2)-3\tan^{-1}0-\frac{1}{2}\ln(1)\right]$

$=-0+\frac{3\pi}{4}+\frac{1}{2}\ln 2+\ln 2-0-0=\frac{3\pi}{4}+\frac{3}{2}\ln 2$ **[M2A1]**

(b) $\int_1^2 x^2\ln x\,dx=\left[\frac{1}{3}x^3\ln x\right]_1^2-\int_1^2\frac{1}{3}x^3\frac{1}{x}dx=\left[\frac{1}{3}x^3\ln x\right]_1^2-\frac{1}{3}\int_1^2 x^2dx$ **[2]**

$=\left[\frac{1}{3}x^3\ln x-\frac{1}{9}x^3\right]_1^2$ **[1]**

$=\frac{8}{3}\ln 2-\frac{8}{9}-0+\frac{1}{9}$ **[1]**

$=\frac{8}{3}\ln 2-\frac{7}{9}$. **[1]**

6(a)(i) $f(x)=2-x-\ln x=0.0945$ when $x=1.5$ and $=-0.0700$ when $x=1.6$. **[2]**

Since the sign changes there must be a root between these points. **[1]**

(ii) $f'(x)=-1-\frac{1}{x}$. **[1]**

$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=1.5-\frac{0.0945}{-1.6667}=1.5567$. **[M1A1]**

$x_2=x_1-\frac{f(x_1)}{f'(x_1)}=1.5567-\frac{0.0007}{-1.6424}=1.5571$ **[M1A1]**

(b) $10\sin^2\theta-3-\sin\theta=0$ **[2]**

$(2\sin\theta+1)(5\sin\theta-3)=0$ **[1]**

$\sin\theta=-\frac{1}{2}$ or $\sin\theta=\frac{3}{5}$. **[2]**

$\theta=0.644, 2.50, 3.67, 5.76$ **[2]**

7(a) $\int \frac{1}{y^2}\mathrm{d}y = \int \sec^2 x \mathrm{d}x$ **[1]**

$-\frac{1}{y} = \tan x + c$ **[2]**

$-1 = \tan\frac{\pi}{4} + c$

$c = -2$ **[1]**

$-\frac{1}{y} = \tan x - 2$

$y = \frac{1}{2-\tan x}$ **[1]**

(b)(i) There are $20-t$ litres of liquid in the tank at time t. **[1]**

So the concentration at time t is $\frac{x}{20-t}$. **[1]**

(ii) Since 3 litres of solution leave the tank each minute the rate of removal is 3 times the concentration. So $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{-3x}{20-t}$. **[2]**

(iii) $\int \frac{\mathrm{d}x}{x} = -3\int \frac{\mathrm{d}t}{20-t}$ **[1]**

$\ln x = 3\ln(20-t) + c$ **[2]**

$x = C(20-t)^3$ **[1]**

(iv) $5 = 8000C$

$C = \frac{5}{8000}$. **[1]**

$t = 10 \Rightarrow x = \frac{5}{8000}(20-10)^3 = \frac{5000}{8000} = \frac{5}{8}(= 0.625)$. **[1]**

8(i) $\overrightarrow{AB} = -2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$, **[1]**

$\overrightarrow{OC}.\overrightarrow{AB} = -4\times 2 - 2\times 4 + 4\times 4 = 0$ **[1]**

So the vectors are perpendicular. **[1]**

(ii) $L_1 = 4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k} + \lambda(4\mathbf{i} - 5\mathbf{j} - \mathbf{k})$ **[2]**

(iii) If L_1, L_2 intersect then

$4 + 4\lambda = 2 + 5\mu \qquad 4\lambda - 5\mu = -2$ **[1]**

$-2 - 5\lambda = -1 - 7\mu \qquad 5\lambda - 7\mu = -1$ **[1]**

$-4 - \lambda = -5 - 2\mu \qquad \lambda - 2\mu = 1$ **[1]**

Solving these we get $\lambda = -3, \mu = -2$ which satisfies all three equations **[2]**

So the lines intersect at $-8\mathbf{i} + 13\mathbf{j} - \mathbf{k}$. **[1]**

(iv) $(4\mathbf{i} - 5\mathbf{j} - \mathbf{k}).(5\mathbf{i} - 7\mathbf{j} - 2\mathbf{k}) = 20 + 35 + 2 = 57$ **[1]**

$|4\mathbf{i} - 5\mathbf{j} - \mathbf{k}| = \sqrt{16+25+1} = \sqrt{42}$

$|5\mathbf{i} - 7\mathbf{j} - 2\mathbf{k}| = \sqrt{25+49+4} = \sqrt{78}$ **[1]**

$\cos\theta = \frac{57}{\sqrt{42}\sqrt{78}} \approx 0.9959$

$\theta = 0.0909 = 5.21°$ **[M1A1]**