Practice Exam 5 (Engineers)

Question 1

The grouped frequency table below shows the number of mobile phones sold per day in a city centre store during a period of 200 working days.

Number of phones sold	Frequency
$66-80$	20
$81-85$	30
$86-90$	55
$91-95$	50
$96-100$	35
$101-115$	10

(i) Calculate the mean number of mobile phones sold per day. **[4]**

(ii) Draw the cumulative frequency curve for the data. **[5]**

(iii) Using the cumulative frequency curve, explain how to obtain the median of the data and find its value. **[3]**

(iv) Find the interquartile range of the data. **[3]**

Question 2

The functions f and g are defined as follows:

$$f(x) = \sin(\pi x) \qquad 0 < x < 2$$
$$g(x) = 1 + x^2 \qquad x \geq 0$$

(i) Write down the range of $f(x)$.

Solve the equation $f(x) = \frac{1}{2}$.

Explain why the function f does not have an inverse. **[4]**

(ii) The inverse of g is g^{-1}. Find $g^{-1}(x)$.

Write down the domain of $g^{-1}(x)$. **[3]**

(iii) Show that $g(|f(x)|) = \dfrac{3-\cos(2\pi x)}{2}$. **[5]**

(iv) Sketch the graph of $g(|f(x)|)$. **[3]**

Question 3

(a) Differentiate $e^x \ln x$ with respect to x. **[2]**

(b) Find the stationary value (turning point) of $y = e^{-x} \cos x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$,

and decide if this is a maximum or a minimum. **[7]**

(c) Find the equation of the normal to the curve $y = (2x-3)^3$

at the point $(2,1)$. **[6]**

Question 4

(a) Use the substitution $u = \ln x$ to evaluate $\int_{e}^{e^2} \frac{1}{x(\ln x)^2} dx$. **[6]**

(b) Let $f(x) = x(3-x)$.

(i) Sketch the curve $y = f(x)$ where $y \geq 0$. **[2]**

(ii) Find the area under this curve in this region. **[3]**

(iii) The area is rotated completely about the x-axis.

Find the exact value of the volume of the solid formed. **[4]**

Question 5

(a) (i) Express $\frac{x+5}{(x^2-1)(x+2)}$ as a sum of partial fractions. **[4]**

(ii) Hence evaluate

$$\int_{2}^{4} \frac{x+5}{(x^2-1)(x+2)} dx.$$ **[6]**

(b) Use integration by parts to find the exact value of $\int_{0}^{\pi/2} x^2 \sin x dx$. **[5]**

Question 6

(a) (i) Show that the equation $\cos x = x$ has a root between 0.7 and 0.8. **[3]**

(ii) Starting with initial value $x_0 = 0.7$,

use the Newton-Raphson method, twice,

to give a better approximation to the root of the equation in (i)

giving your final answer correct to 4 decimal places. **[5]**

(b) Solve $4\cos 2\theta + 1 = 2\cos\theta$ for $0 < \theta < 2\pi$. **[7]**

Question 7

(a) Solve the differential equation

$$y\frac{\mathrm{d}y}{\mathrm{d}x} = 4\sin x$$

given that $y = 1$ when $x = 0$. Express your answer in the form $y = f(x)$. **[5]**

(b) In an epidemic the number of people infected, y,

decreases as the method of treatment improves.

The number decreases at a rate proportional to the product of y

and the time t since the treatment was started.

(i) Express this information as a differential equation for $t \geq 0$. **[2]**

(ii) Find the general solution of the equation. **[5]**

(iii) If initially 2000 were infected

and after 20 days the number had reduced to 1000,

find how many were still infected after a further 20 days. **[3]**

Question 8

Let A, B and C be points with position vectors

$$\mathbf{a} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k},\ \mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k},\ \mathbf{c} = 3\mathbf{i} - 2\mathbf{j} + 3\mathbf{k},$$

respectively, with respect to the origin O.

(i) Show that $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$. **[3]**

(ii) Let L_1 be the line passing through C parallel to $\overrightarrow{OA}$.

Find the vector equation of L_1. **[2]**

(iii) Show that L_1 intersects L_2,

where $L_2 = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + \mu(2\mathbf{i} - 5\mathbf{j} + 6\mathbf{k})$,

and find the point of intersection. **[6]**

(iv) Find the acute angle between the lines L_1 and L_2. **[4]**

Practice Exam 5 Answers

1(i)

		x	f	xf
65.5	80.5	73	20	1460
80.5	85.5	83	30	2490
85.5	90.5	88	55	4840
90.5	95.5	93	50	4650
95.5	100.5	98	35	3430
100.5	115.5	108	10	1080
			200	17950

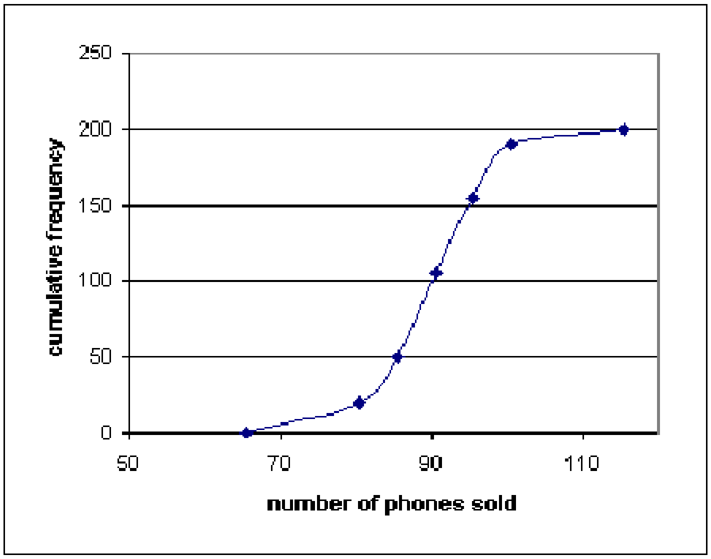
x column **[1]**

xf column **[2]**

mean $= \dfrac{\sum xf}{\sum f} = \dfrac{17950}{200} = 89.75$. **[1]**

(ii)

lcb	ucb	cf
	65.5	0
65.5	80.5	20
80.5	85.5	50
85.5	90.5	105
90.5	95.5	155
95.5	100.5	190
100.5	115.5	200



see above

labels on vertical axis either absolute or as percentage. **[1]**

values plotted on upper class boundaries **[1]**

points accurately found and plotted **[2]**

smooth curve **[1]**

(iii) On graph look at the 50% line, labelled 100, and read down. **[2]**

The value is 90. (Allow variation according to graph drawn.) **[1]**

(iv) The lower quartile is 85.5, (Allow variation according to graph drawn.) **[1]**

the upper quartile is 95, (Allow variation according to graph drawn.) **[1]**

the difference is 9.5. (Allow variation according to graph drawn.) **[1]**

2(i) The range is $-1 \le y \le 1$. **[1]**

$x = \dfrac{1}{6}$ or $\dfrac{5}{6}$. (Accept 0.167 or 0.833.) **[2]**

The function is not one-one, as we have just established. **[1]**

(ii) $g^{-1}(x) = \sqrt{x-1}$ **[2]**

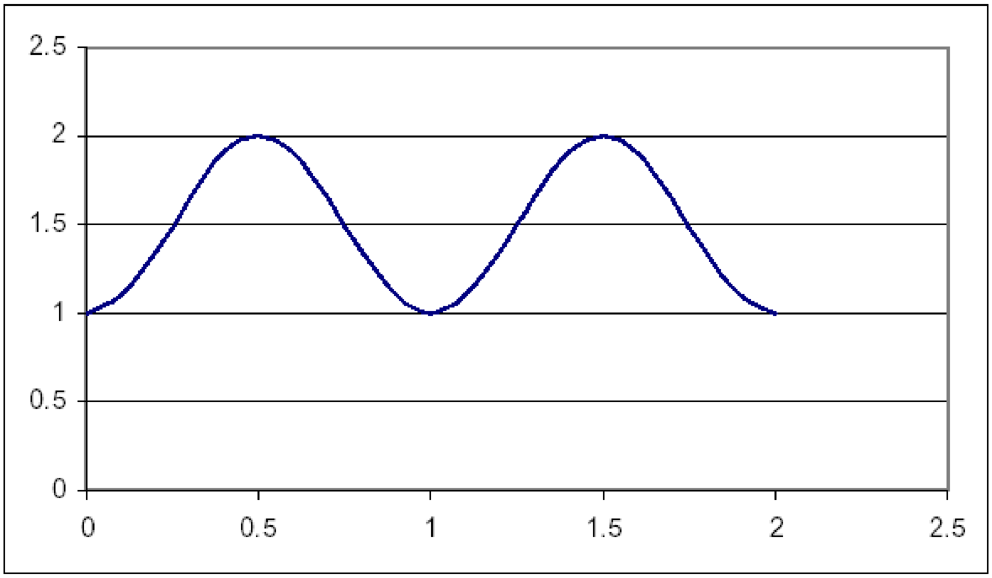
Its domain is $1 \le x < \infty$ or simply $1 \le x$. **[1]**

(iii) $g(f(x)) = 1 + (f(x))^2 = 1 + \sin^2(\pi x)$ **[2]**

$= 1 + \dfrac{1}{2}(1 - \cos(2\pi x))$ **[2]**

$= \dfrac{3 - \cos(2\pi x)}{2}$ **[1]**

(iv)



Shape	**[1]**
position on x-axis	**[1]**
position on y-axis	**[1]**

3(a) $\frac{dy}{dx} = e^x \ln x + e^x \frac{1}{x}$ **[2]**

(b) $\frac{dy}{dx} = -e^{-x}\cos x - e^{-x}\sin x$ **[1]**

$= 0$ at a stationary point. **[1]**

Then $\cos x = -\sin x$, $\tan x = -1$, $x = -\frac{\pi}{4}(\approx -0.785)$. **[1]**

Here $y = e^{\pi/4}\frac{1}{\sqrt{2}}(\approx 1.55)$. **[1]**

$\frac{d^2y}{dx^2} = e^{-x}\cos x + e^{-x}\sin x + e^{-x}\sin x - e^{-x}\cos x = 2e^{-x}\sin x$.

When $x = -\pi/4$, $\frac{d^2y}{dx^2} = 2e^{\pi/4}\sin(-\pi/4) = -\sqrt{2}\,e^{\pi/4} < 0$. **[2]**

So the stationary value is a maximum. **[1]**

(c) $\frac{dy}{dx} = 3(2x-3)^2 2 = 6(2x-3)^2$ **[1]**

$= 6$ at the point $(2,1)$. **[1]**

The gradient of the normal is therefore $-\frac{1}{6}$. **[1]**

The equation of the normal is $y = -\frac{1}{6}x + c$. **[1]**

$1 = -\frac{1}{6}\times 2 + c$, $c = \frac{4}{3}$. **[1]**

$y = -\frac{1}{6}x + \frac{4}{3}$.

$x + 6y = 8$. **[1]**

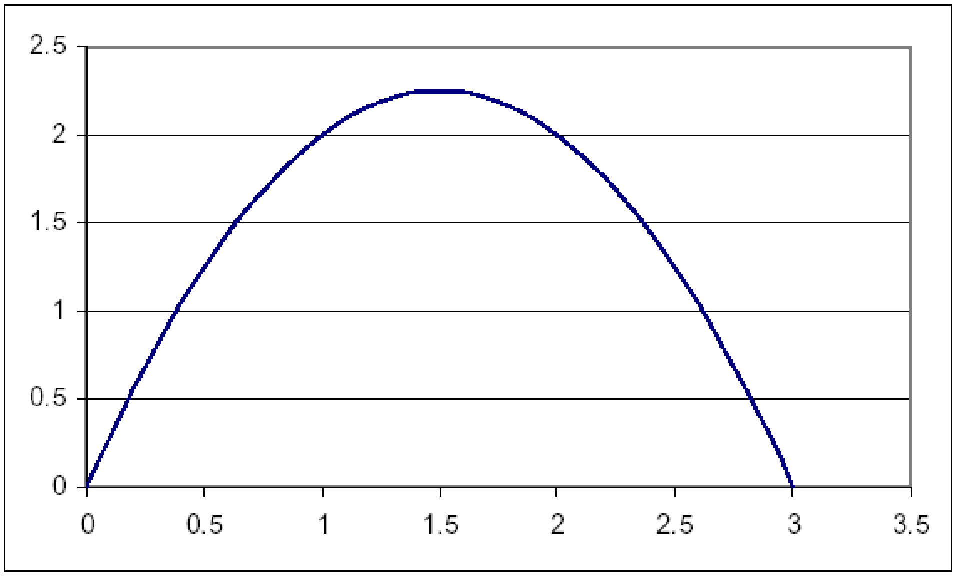
4(a) $u = \ln x,\ x = e^u,\ du = \dfrac{dx}{x}$. **[2]**

$\displaystyle\int_e^{e^2} \frac{1}{x(\ln x)^2}\,dx = \int_1^2 \frac{1}{u^2}\,du$ **[2]**

$= \left[-\dfrac{1}{u}\right]_1^2$ **[1]**

$= -\dfrac{1}{2} + 1 = \dfrac{1}{2}$. **[1]**

(b)(i)



Shape **[1]**
limits **[1]**

(ii) $\displaystyle\int_0^3 (3x - x^2)\,dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$ **[1]**

$= \left[\dfrac{27}{2} - \dfrac{27}{3}\right]$ **[1]**

$= \dfrac{9}{2}$ **[1]**

(iii) $\displaystyle\pi\int_0^3 (3x - x^2)^2\,dx = \pi\int_0^3 (9x^2 - 6x^3 + x^4)\,dx$ **[1]**

$= \pi\left[\dfrac{9x^3}{3} - \dfrac{6x^4}{4} + \dfrac{x^5}{5}\right]_0^3$ **[1]**

$= \pi\left[81 - \dfrac{243}{2} + \dfrac{243}{5}\right]$ **[1]**

$= \dfrac{81\pi}{10}(\approx 25.4)$. **[1]**

5(a)(i) $\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x+2}=\frac{A(x+1)(x+2)+B(x-1)(x+2)+C(x-1)(x+1)}{(x^2-1)(x+2)}$. **[1]**

This equals $\frac{x+5}{(x^2-1)(x+2)}$

$x=1 \Rightarrow 6A=6 \Rightarrow A=1$. **[1]**

$x=-1 \Rightarrow -2B=4 \Rightarrow B=-2$. **[1]**

$x=-2 \Rightarrow 3C=3 \Rightarrow C=1$. **[1]**

Given expression equals $\frac{1}{x-1}-\frac{2}{x+1}+\frac{1}{x+2}$.

(ii) Given integral equals

$\int_2^4\left(\frac{1}{x-1}-\frac{2}{x+1}+\frac{1}{x+2}\right)dx=\left[\ln(x-1)-2\ln(x+1)+\ln(x+2)\right]_2^4$ **[2]**

$=\left[\ln 3-2\ln 5+\ln 6-\ln 1+2\ln 3-\ln 4\right]$

$=\left[(1+1+2)\ln 3-2\ln 5+(1-2)\ln 2\right]$

$=4\ln 3-2\ln 5-\ln 2=\ln\left(\frac{81}{50}\right)(\approx 0.482)$. **[M2A2]**

(b) $\int_0^{\pi/2}x^2\sin x\,dx=\left[-x^2\cos x\right]_0^{\pi/2}+\int_0^{\pi/2}2x\cos x\,dx$ **[1]**

$=\left[-x^2\cos x+2x\sin x\right]_0^{\pi/2}-\int_0^{\pi/2}2\sin x\,dx$ **[1]**

$=\left[-x^2\cos x+2x\sin x+2\cos x\right]_0^{\pi/2}$ **[1]**

$=0+\pi+0+0-0-2$ **[1]**

$=\pi-2$. **[1]**

6(a)(i) $f(x)=\cos x-x=0.0648$ when $x=0.7$ and $=-0.1033$ when $x=0.8$. **[2]**

Since the sign changes there must be a root between these points. **[1]**

(ii) $f'(x)=-\sin x-1$. **[1]**

$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=0.7-\frac{0.0648}{-1.6442}=0.7394$. **[M1A1]**

$x_2=x_1-\frac{f(x_1)}{f'(x_1)}=0.7394-\frac{-0.00059}{-1.6739}=0.7391$ **[M1A1]**

(b) $8\cos^2\theta-2\cos\theta-3=0$ **[2]**

$(2\cos\theta+1)(4\cos\theta-3)=0$ **[1]**

$\cos\theta=-\frac{1}{2}$ or $\cos\theta=\frac{3}{4}$. **[2]**

$\theta=0.723, 2.09, 4.19, 5.56$ **[2]**

7(a) $\int y\,dy = \int 4\sin x\,dx$ **[1]**

$\frac{y^2}{2} = -4\cos x + c$ **[2]**

$\frac{1}{2} = -4 + c$

$c = \frac{9}{2}$ **[1]**

$\frac{y^2}{2} = -4\cos x + \frac{9}{2}$

$y = \sqrt{9 - 8\cos x}$ **[1]**

(b)(i) $\frac{dy}{dt} = -kyt$ **[2]**

(ii) $\int \frac{dy}{y} = \int -kt\,dt$ **[1]**

$\ln y = -\frac{kt^2}{2} + c$ **[2]**

$y = C\,e^{-kt^2/2}$ **[2]**

(iii) When $t = 0,\ y = C$.

$1000 = 2000\,e^{-200k}$ **[1]**

$e^{200k} = 2$

$200k = \ln 2$

$k = \frac{\ln 2}{200} (\approx 0.003466)$. **[1]**

$t = 40 \Rightarrow y = 2000\,e^{-4\ln 2} = 2000/16 = 125$

Alternatively, $t = 40 \Rightarrow y \approx 2000\,e^{-800\times 0.003466} \approx 125$ **[1]**

8(i) $\overrightarrow{AB} = \mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$, **[1]**

$\overrightarrow{OC}.\overrightarrow{AB} = 3\times 1 - 2\times 6 + 3\times 3 = 0$ **[1]**

So the vectors are perpendicular. **[1]**

(ii) $L_1 = 3\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})$ **[2]**

(iii) If L_1, L_2 intersect then

$3 + \lambda = 2 + 2\mu \qquad \lambda - 2\mu = -1$ **[1]**

$-2 - 3\lambda = 3 - 5\mu \qquad 3\lambda - 5\mu = -5$ **[1]**

$3 + 2\lambda = 5 + 6\mu \qquad 2\lambda - 6\mu = 2$ **[1]**

Solving these we get $\lambda = -5,\ \mu = -2$ which satisfies all three equations **[2]**

So the lines intersect at $-2\mathbf{i} + 13\mathbf{j} - 7\mathbf{k}$. **[1]**

(iv) $(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}).(2\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}) = 2 + 15 + 12 = 29$ **[1]**

$|\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}| = \sqrt{1+9+4} = \sqrt{14}$

$|2\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}| = \sqrt{4+25+36} = \sqrt{65}$ **[1]**

$\cos\theta = \frac{29}{\sqrt{14}\sqrt{65}} \approx 0.9613$

$\theta = 0.279 = 16.0°$ **[M1A1]**