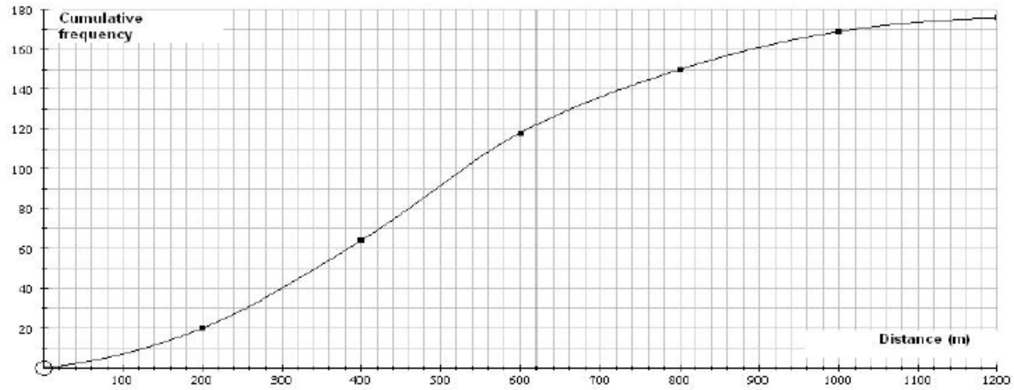
Practice Exam 3 (Engineers)

1 The cumulative frequency graph below illustrates the distance that 176 students live from their university.



Use the graph to estimate to the nearest 10 metres:

(i) the median distance from school; **[1]**

(ii) the lower quartile, the upper quartile and the inter-quartile range. **[3]**

The graph used the following cumulative frequency grouped data:

Distance (metres)	200	400	600	800	1000	1200
Cumulative frequency	20	64	118	150	169	176

(iii) Copy and complete the grouped frequency table below for this data **[3]**

Distance (d metres)	Frequency (f)
$0 < d \leq 200$	20
$200 < d \leq 400$	44
….	….

(iv) Estimate the mean distance using your table. **[4]**

(v) Estimate the standard deviation **[4]**

2 The function f is defined by $f(x) = 1 + \sqrt{x}$ $x \geq 0$,
and the function g is defined by $g(x) = x^2$ $x \in R$.

(i) Find the domain of the inverse function f^{-1}. **[2]**

(ii) Find an expression for $f^{-1}(x)$. **[3]**

(iii) Sketch the graphs of $f(x)$ and $f^{-1}(x)$, using the same axes. **[4]**

(iv) Find and simplify an expression for $fg(x)$ when $x \geq 0$. **[2]**

(v) Explain clearly why the value of $fg(-2)$ is 3. **[1]**

(vi) Sketch the graph of $y = fg(x)$, for both positive and negative values of x, and give the equation of this graph in simplified form. **[3]**

3 (a) Differentiate

$$x^3 \cos 3x \qquad \text{with respect to } x.$$ **[2]**

(b) Find the stationary value (turning point) of **[5]**

$$y = \tan x - 8\sin x \qquad \text{for} \qquad 0 < x < \frac{\pi}{2}.$$

(c) Find the equation of the tangent to the curve **[4]**

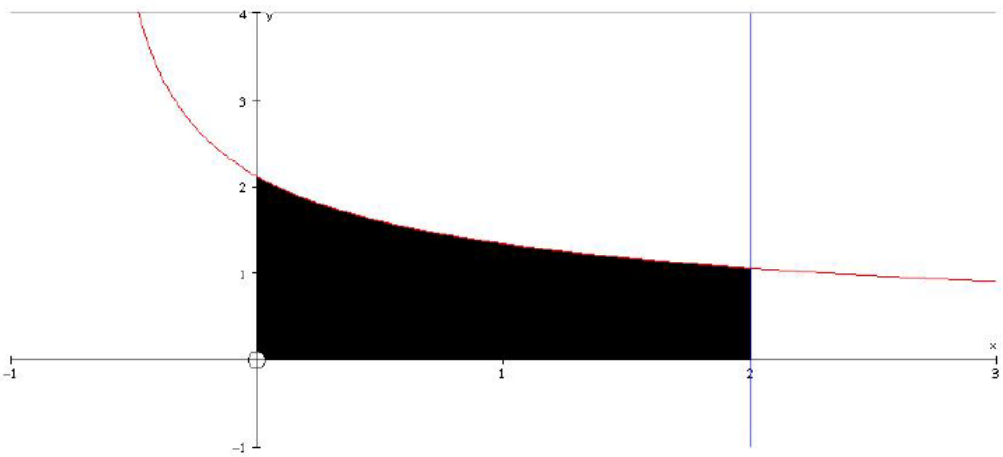
$$y = (x^2+1)^4 \qquad \text{at the point } (1,16).$$

(d) Given that $x = \cos y$, prove that $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{(1-x^2)}}$. **[4]**

4 (a) Use the substitution $u^2 = 1 + x$ to show that **[7]**

$$\int_0^1 \frac{x}{\sqrt{(1+x)}}\,dx = \frac{2}{3}(2-\sqrt{2}).$$

(b)



The diagram above shows the curve

$$y = \frac{3}{\sqrt{(3x+2)}}$$

The shaded region is enclosed by the curve, the two co-ordinate axes and the line $x = 2$.

(i) Show that the exact area of the shaded region is $2\sqrt{2}$. **[4]**

(ii) The shaded region is rotated completely about the x-axis. Find the exact value of the volume of the solid formed. **[4]**

5 (a) By finding values of A, B and C so that **[5]**

$$\frac{x^2}{x^2-1} = A + \frac{B}{x-1} + \frac{C}{x+1}$$

find $\displaystyle\int \frac{x^2}{x^2-1}\,dx$. **[4]**

(b) Use integration by parts to evaluate the integral **[6]**

$$\int_0^{\frac{\pi}{6}} x\cos 3x\,dx.$$

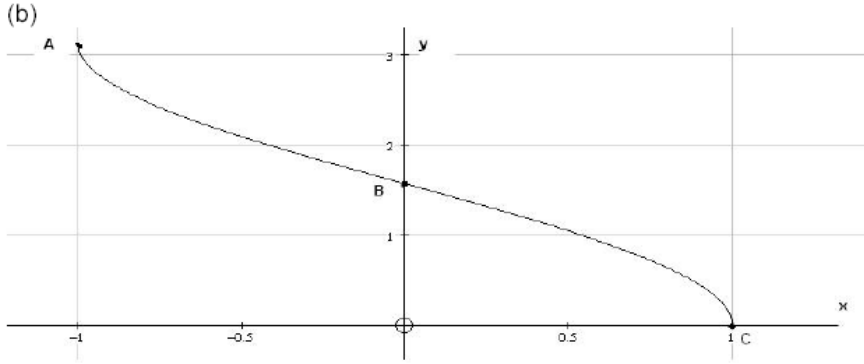
6 (a) Prove the identity **[3]**

$$\sin(x+30^o)+\sqrt{3}\cos(x+30^0) = 2\cos x$$

where x is measured in degrees.

Hence express $\cos 15^0$ in surd form. **[3]**

(b)



The diagram above shows the curve $y = \cos^{-1} x$.

(i) State the coordinates of the points $A:(-1,a)$, $B:(0,b)$ and $C:(1,c)$ **[2]**

(ii) Find the value of x for which $\cos^{-1} x = b/3$. **[2]**

(iii) The equation of a second curve is $y = \frac{1}{2}\cos^{-1}(x-1)$. **[3]**

Describe the geometrical transformation to obtain this curve from $y = \cos^{-1} x$.

(iv) Verify by calculation that the value of x at the point of intersection of the curves is 0.5. **[2]**

7 (a) Solve the differential equation $\frac{dy}{dx} = y^2 e^{-2x}$, given that $y = 1$ when $x = 0$. **[4]**

Express your answer in the form $y = f(x)$.

(b) A model to estimate the value of a car assumes the rate of decrease of V at time t is proportional to V. The initial value is £20,000.

(i) Set up, and solve, a differential equation to show that $V = 20000e^{-kt}$ where k is a positive constant. Show all your working. **[6]**

(ii) Given that the car decreases in value to £12000 in two years, find the value of the car after one year. Give your answer to the nearest £10. **[3]**

(iii) Find the age of the car, in years, when the value is £2000 **[2]**

8 Referred to a fixed origin O, the points A and B have position vectors 3**i** + 2**j** + 4**k** and 4**i** + 4**j** - 3**k**, respectively.

The line L_1 passes through A and has equation

r = (3**i** + 2**j** + 4**k**) + μ (5**i** – 1**j** – 1**k**)

(i) Show that $\overrightarrow{AB}$ = is perpendicular to L_1. **[5]**

(ii) The line L_2 passes through B and has direction vector (2**i** + 1**j** – 2**k**) **[1]**

Find an equation of the line L_2.

(iii) Show that the lines L_1 and L_2 intersect and find the coordinates of their point of intersection. **[6]**

(iv) Calculate the acute angle, in degrees, between L_1 and L_2 . **[3]**

Practice Exam 3 Answers

1

(a) Use the graph to estimate to the nearest 10 metres:

(i) the median distance from school: accept answers 480-500 **[1]**

(ii) the lower quartile, accept answers 300-320 **[1]**

the upper quartile : accept answers 650-670 **1**

the interquartile range. Follow through from the above **1**

(iii)

Distance (d metres)	Frequency (f)
$0 < d \leq 200$	20
$200 < d \leq 400$	44
$400 < d \leq 600$	54
$600 < d \leq 800$	32
$800 < d \leq 1000$	19
$1000 < d \leq 1200$	7
	Sum = 176

[3]

(iv) Estimate the mean distance $= \sum f.(mid-point)/\sum f$ **M1**

= (20 x 100 + 44 x 300 + 54 x 500 + 32 x 700 + 19 x 900 + 7 x 1100)/176 **2**

= 507.95 = 508 (3 s.f.) **1**

(v) Estimate the standard deviation = $\sqrt{\sum fd^2/176 - mean^2}$ **M1**

$\sum fd^2 =$ 57200000 **1**

s.d.^{2} = 32500 – 507.95^2 = 66986.8 **1**

s.d. = 258.8 = 259 (3 s.f.) **1**

2 The function f is defined by $f(x) = 1 + x^{\frac{1}{2}}$ $x \geq 0$,

and the function g is defined by $g(x) = x^2$ $x \in R$.

(i) Domain of f^{-1} is the range of $f(x)$:

$f^{-1}(x) \geq 1$

[2]

(ii) y = 1+ √x

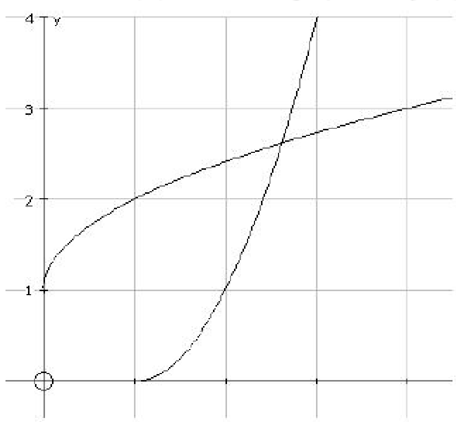
√x = y – 1 **1**

x = (y – 1)^2

$f^{-1}(x) = (x-1)^2$ (or x^2 -2x +1) **1**

1

(iii)Sketch the graphs of $f(x)$ and $f^{-1}(x)$, using the same axes.



s
2

Shape
2

Find and simplify an expression for $fg(x)$ when $x \geq 0$.

$f(x^2)$ **1**

$= 1 + x \qquad x \geq 0$ **1**

Explain clearly why the value of $fg(-2)$ is 3.

$f(4)$ **1**

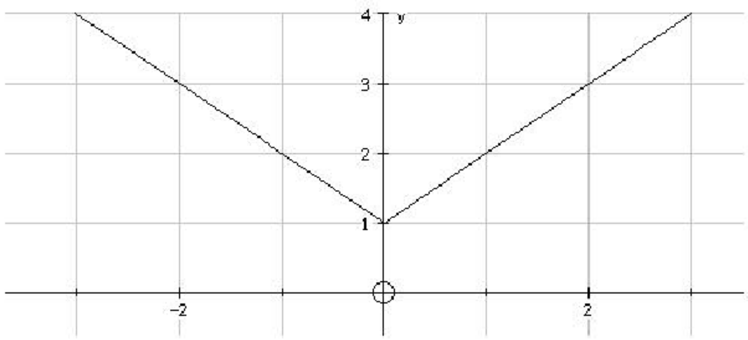
$= 1 + \sqrt{4}$ **1**

$= 3$

Sketch the graph of $y = fg(x)$, for both positive and negative values of x, and give the equation of this graph in simplified form.

$f(x^2) \qquad x \in R$

$= 1 + |x| \qquad x \in R$ **1**



2 (1 for each branch)

3a

$$\frac{d}{dx}(x^3 \cos 3x) = 3x^2 \cos 3x - 3x^3 \sin 3x$$ 2

b

$$y = \tan x - 8 \sin x \qquad 0 < x < \frac{\pi}{2}$$

$$\frac{dy}{dx} = \sec^2 x - 8 \cos x$$ 1

$$\frac{1}{\cos^2 x} - 8 \cos x = 0$$ 1

$$\frac{1 - 8\cos^3 x}{\cos^2 x} = 0$$

$$8\cos^3 x = 1 \qquad \cos x \neq 0$$ 1

$$\cos x = \tfrac{1}{2}$$

$$x = \tfrac{\pi}{3} \qquad y = \sqrt{3} - 8\left(\frac{\sqrt{3}}{2}\right) = -3\sqrt{3}$$ 1 1

c

$$y = (x^2 + 1)^4 \quad \text{at the point } (1, 16).$$

$$\frac{dy}{dx} = 4(x^2 + 1)^3 2x = 8x(x^2 + 1)^3$$ 2

When $x = 1, \frac{dy}{dx} = 64$ 1

Equation of the tangent at (1,16): y-16 = 64(x-1)

y = 64x -48 1

d

Given that $x = \cos y$ prove that $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{(1 - x^2)}}$

$$\frac{dx}{dy} = -\sin y$$ 1

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$ 1

$$\frac{dy}{dx} = -\frac{1}{\sqrt{(1 - \cos^2 y)}}$$ 1

$$\frac{dy}{dx} = -\frac{1}{\sqrt{(1 - x^2)}}$$ 1

4

$u^2 = 1 + x$ 1

$\frac{dx}{du} = 2u$ 1

When x=0, u=1 and at x=1, u=$\sqrt{2}$ 1

$\int_0^1 \frac{x}{\sqrt{(1+x)}}dx = \int_1^{\sqrt{2}} \frac{(u^2-1)}{u}2u\,du$ 1

$= \int_1^{\sqrt{2}} (2u^2 - 2)du$ 1

$= \left[\frac{2u^3}{3} - 2u\right]_1^{\sqrt{2}}$ 1

$= \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2}\right) - \left(\frac{1}{3} - 1\right)$ 1

$= \frac{4}{3} - \frac{2\sqrt{2}}{3} = \frac{2}{3}(2 - \sqrt{2})$ 1

(b) (i) Area = $\int_0^2 \frac{3}{\sqrt{(3x+2)}}dx$ 1

$= \int_0^2 3(3x+2)^{-1/2}dx$ 1

$= \left[\frac{3}{3.(1/2)}(3x+2)^{1/2}\right]_0^2$ 1

$= 2\sqrt{8} - 2\sqrt{2}$ 1

$= 4\sqrt{2} - 2\sqrt{2} = 2\sqrt{2}$

(ii) Volume = $\int_0^2 \pi y^2 dx$ 1

$= \pi\int_0^2 \frac{9}{3x+2}dx$ 1

$= \left[\frac{9\pi}{3}(\ln|3x+2|)\right]_0^2$ 1

$= 3\pi(\ln 8 - \ln 2) = 3\pi \ln 4$ 1

5

$\frac{x^2}{x^2-1} = A + \frac{B}{x-1} + \frac{C}{x+1}$ 1

$x^2 = A(x^2-1) + B(x+1) + C(x-1)$

Let x=1 $\Rightarrow$ B=1/2 1

Let x=0 $\Rightarrow$ A + C = ½ 1

Let x = 2 $\Rightarrow$ 3A + C = 5/2

Subtract; 2A = 2, A = 1

Substitute; C = -1/2 1

$$\frac{x^2}{x^2-1} = 1+\frac{1}{2(x-1)}-\frac{1}{2(x+1)}$$

$$\int\frac{x^2}{x^2-1}dx = \int 1+\frac{1}{2(x-1)}-\frac{1}{2(x+1)}dx$$

$$x+\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1| + c$$ 3

$$= x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + c$$ 1

(b) $\int x\cos 3x\,dx$

1

Let u = x and dv/dx = cos 3x

$$v = \frac{\sin 3x}{3}$$ 1

$$\int x\cos 3x\,dx = \left[x\frac{\sin 3x}{3}\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}}\frac{\sin 3x}{3}dx$$ 2

$$= \left[\frac{x}{3}\sin 3x+\frac{\cos 3x}{9}\right]_0^{\frac{\pi}{6}}$$ 2

$$= \frac{\pi}{18}+0-(0+1/9)$$

$$=\frac{\pi}{18}-1/9$$ 1

6a

$$\sin(x+30^o)+\sqrt{3}\cos(x+30^0) = \sin x\cos 30+\cos x\sin 30+\sqrt{3}\cos x\cos 30-\sqrt{3}\sin x\sin 30$$ 1 1

$$=\frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x+\frac{3}{2}\cos x-\frac{\sqrt{3}}{2}\sin x$$ 1

$$= 2\cos x$$

$2\cos 15^0$ = sin 45 + $\sqrt{3}$ cos45 1

= $1/\sqrt{2}+\sqrt{3}/\sqrt{2}$ 1

cos 15^0 = $\frac{1+\sqrt{3}}{\sqrt{2}}$ 1

(b) (i) A:(-1, π), B:(0, $\pi/2$) and C: (1,0) 1 0.5 0.5

(ii) $\cos^{-1}x = \pi/6$, x=cos($\pi/6$), x= $\sqrt{3}/2$ 1 1

(iii) .

Translation, 1 unit along the x axis, 2

Stretch, s.f. 0.5, along the y axis. 2

(iv) y=$\cos^{-1}$ 0.5 = $\pi/3$ 1

$y=\frac{1}{2}\cos^{-1}(x-1)$ = 0.5 $\cos^{-1}$ (-0.5) = 0.5 ($\frac{2\pi}{3}$) = $\pi/3$ 1

7 a) $\int \frac{1}{y^2} dy = \int e^{-2x} dx$, separating the variables 1

$-y^{-1} = \frac{e^{-2x}}{-2} + c$ 1

y=1 when x=0 $\Rightarrow c = -1/2$ 1

$-\frac{1}{y} = \frac{e^{-2x}}{-2} + \frac{1}{-2}$

$y = \frac{2}{e^{-2x} + 1}$ 1

(a)

(i) $\frac{dV}{dt} = -kV$ 1

$\int \frac{1}{V} dV = \int -k dt$ 1

$\ln V = -kt + c$ 1

t=0, V=20000, c = ln 20000 1

$\ln \frac{V}{20000} = -kt$ 1

$V = 20000e^{-kt}$ 1

(ii) t=2, V=12000; ln(12000/20000) = -2k 1 1

k=0.255 1

t=1, V=20000 e $^{-0.255}$ = £15490 (nearest £10) (Accept +/- £10) 1

(iii) Find the age of the car when the value was £2000

ln (2000/ 20000) = -0.255 t t= 9 years 1,1

8 Referred to a fixed origin O, the points A and B have position vectors 3**i** + 2**j** + 4**k** and 4**i** + 4**j** - 3**k** respectively.
The line L_1 passes through A and has equation

r = (3**i** + 2**j** + 4**k**) + μ (5**i** + 1**j** + 1**k**) .

(i) Show that $\overrightarrow{AB}$ is perpendicular to L_1 .

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

= (4**i** + 4**j** - 3**k**) – (3**i** + 2**j** + 4**k**) **M1**

= **i** + 2**j** -7**k** **A1**

$\overrightarrow{AB}$ **.** (5**i** + 1**j** + 1**k**) = (**i** + 2**j** -7**k**) **.** (5**i** + 1**j** + 1**k**) = 5 + 2 - 7 = 0 **M1**

$\therefore$ **OA** is perpendicular to **AB** **A1** **1**

(ii) The line L_2 passes through B and has direction vector (2**i** + 1**j** – 2**k**)

Find an equation of the line L_2 . **[1]**

r = (4**i** + 4**j** - 3**k**) + λ (2**i** + 1**j** – 2**k**)

(iii) Show that the lines L_1 and L_2 intersect and find the coordinates of their point of intersection.

Equating the x and y components of L_1 *and* L_2;

4 + 2λ **=** 3 + 5 μ

4 + λ **=** 2 + 1 μ **1**

λ **=** -3, μ = -1 **1** **2**

Check the z component for consistency:

4 + μ = 3, -3 - 2λ = -3 + 6 = 3 **1**

Point of intersection (-2, 1, 3) **1**

(iv) Calculate the acute angle, to the nearest degree, between L_1 and L_2 .

(5**i** + 1**j** + 1**k**) **.** (2**i** + 1**j** – 2**k**) = $\sqrt{27}\sqrt{9}\cos\theta$ **M1**

$\cos\theta = 9/9\sqrt{3}$ **A1**

$\theta = 54.7^0$ **1**