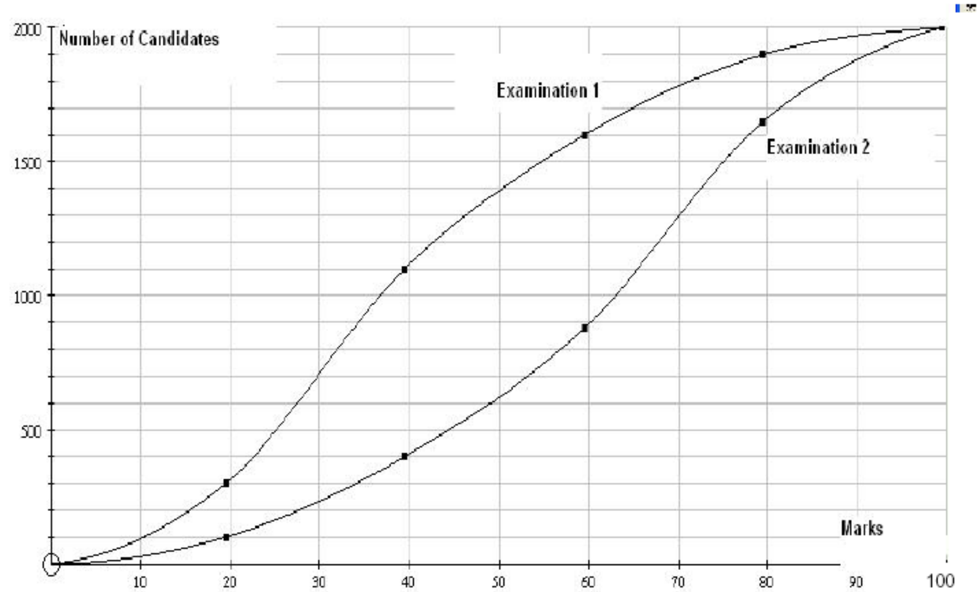
Practice Exam 4 (Engineers)

1 The diagram below shows the cumulative frequency graphs for the marks scored by the candidates in two examinations. The 2000 candidates each took 2 examinations.



(i) Use the diagram to estimate the median mark for examination 1 and examination 2, and the inter-quartile range for examination 1. **[5]**

(ii) State, with a reason, which of the two examinations you think was the easier one. **[2]**

(iii) Copy and complete the grouped frequency table shown below for examination one and estimate the mean and standard deviation of the marks for examination 1. **[8]**

Marks	Cumulative frequency	Frequency
0 - 19	300	
20 - 39		
…		

2 The functions f and g are defined by

$$f(x) = \frac{4}{3+x}, \quad x > 0$$

$$g(x) = 9 - x^2, \quad x \in \mathbb{R}.$$

a) Find $fg(x)$. **[3]**

b) (i) Solve the equation $g(x) = 5$ **[2]**
(ii) Explain why the function g does not have an inverse. **[1]**

c) Sketch the graph of $y = g(x)$. **[2]**
Using the same axes, sketch the graph $y = |g(x)|$. **[2]**

d) (i) Find the range of $f(x)$, where $f(x)$ is the function above. **[2]**

(ii) The inverse of f is f^{-1}. Find $f^{-1}(x)$. **[3]**

3 (a) Differentiate xe^{2x} with respect to x. **[2]**

(b) Given $y = (\ln x)^2$ $\quad x > 0$

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. **[3]**

(ii) Find the coordinates of the point P at which the *gradient* takes its maximum value. **[4]**

(c) (i) Given that $y = \tan^{-1} x$, express x in terms of y. **[4]**

Find $\frac{dx}{dy}$ in terms of y.

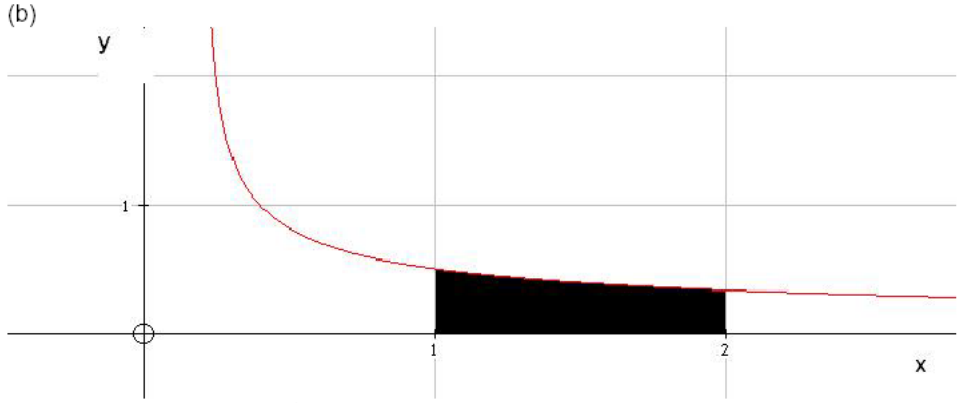
Show that $\frac{dy}{dx} = \frac{1}{1+x^2}$.

(ii) Evaluate $\int_0^1 \frac{1}{1+x^2} dx$ **[2]**

4 (a) Use the substitution $u = \sin x$ to evaluate **[6]**

$$\int_{\pi/6}^{\pi/2} \cos 2x \cos x \, dx .$$

(b)



The diagram above shows the curve

$$y = \frac{1}{\sqrt{(5x-1)}} .$$

The shaded region is enclosed by the curve, the x-axis, the line $x = 1$ and the line $x = 2$.

(i) Show that the exact value of the area of the shaded region is $2/5$. **[5]**

(ii) The shaded region is rotated completely about the x-axis. Find the exact value of the volume of the solid formed. **[4]**

5 (a) (i) Express $\frac{2}{x(x+1)(x+2)}$ as the sum of partial fractions **[4]**

(ii) Hence show that $\int_2^4 \frac{2}{x(x+1)(x+2)} dx = 2\ln 3 - 2\ln 5$ **[4]**

(b) (i) Using the identity $\tan x \equiv \frac{\sin x}{\cos x}$, $\quad -\frac{\pi}{2} < x < \frac{\pi}{2}$, **[3]**

show that $\frac{d}{dx}(\tan x) = \sec^2 x$.

(ii) Use integration by parts to find **[4]**

$$\int x \sec^2 x \, dx$$

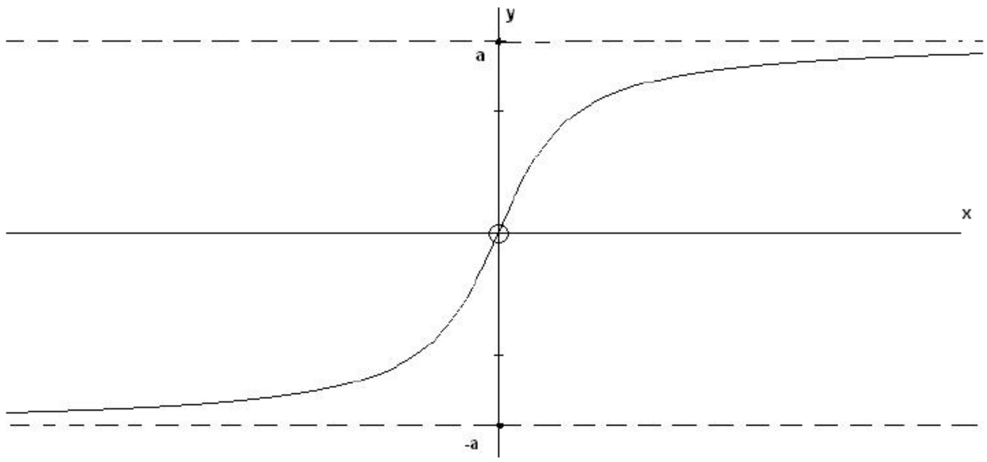
6 (a)

(i) Given that $\sin(x+\alpha) = 2\sin(x-\alpha)$, show that $\tan x = 3\tan\alpha$. **[3]**

(ii) Use the result of part (i) to solve the equation **[3]**

$$\sin\left(2y+\frac{\pi}{4}\right)=2\sin\left(2y-\frac{\pi}{4}\right) \quad \text{for} \quad 0 < y < \pi$$

giving your answer in radians.



(b) The diagram above shows the curve $y = \tan^{-1} x$ and its asymptotes $y = \pm a$.

(i) State the exact value of a. **[1]**

(ii) Find the value of x for which $\tan^{-1} x = \frac{1}{2}a$. **[2]**

(iii) The equation of a second curve is $y = 2\tan^{-1}(x-1)$. Describe the geometrical transformations to obtain this curve from $y = \tan^{-1} x$. **[4]**

(iv) Verify, by calculation that the value of x at the point of intersection of the two curves, $y = \tan^{-1} x$ and $y = 2\tan^{-1}(x-1)$, is 1.54, correct to 2 decimal places. **[2]**

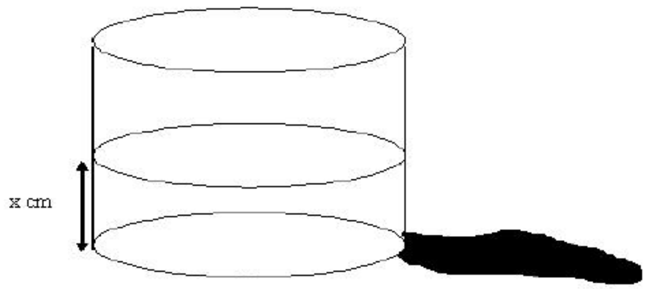
7 (a) .

(i) Solve the differential equation $\frac{dx}{dt} = \frac{20 - x}{10}$, given that $x = 5$ when $t = 0$. **[5]**

Express x as a function of t.

(ii) Find the value of t for which $x = 10$. **[2]**

(b) A cylindrical container has a height of 400 cm. The container was initially full of a chemical but there is a leak from a hole in the base.



(i) It can be assumed that when the depth of the chemical remaining is x cm, the rate at which the level is **dropping**, $\frac{dx}{dt}$, is proportional to $\sqrt{x}$. **[3]**

Set up and solve an appropriate differential equation.

(ii) Initially the container was full, and the depth of the chemical was x = 400cm. **[3]**
When the container was half full, the level was dropping at a rate of 1 cm per minute.
Find the constants in the equation.

(iii) When the container was half full, show that the container had been leaking for more than $2\frac{1}{2}$ hours. **[2]**

8 (a) The position vectors of 3 points A, B, C are **[5]**
a = 4**i** + 2**j** - **k** , **b** = -2**i** +26**j** + 11**k** , **c** = 16**i** + 17**j** + 2**k**

Show that vector 2**i** - 3**j** + 7**k** is perpendicular to $\overrightarrow{AB}$ and also perpendicular to $\overrightarrow{AC}$.

(b) The vector equations of two straight lines are

r = 5**i** +3**j** -2**k** + λ (**i** – 2**j** + 2**k**) and
r = 2**i** - 11**j** +a**k** + μ (-3**i** – 4**j** + 5**k**)

Given that the two lines intersect, find:

(i) the coordinates of the point of intersection; **[5]**

(ii) the value of the constant a; **[2]**

(iii) the acute angle between the lines. **[3]**

Practice Exam 4 Answers

1 (i) The median mark for examination 1 : 38 (+/- 1) **[1]**
median examination 2: 63 (+/- 1) **1**
Examination 1: UQ; 56 (+/- 1) **1**
LQ ; 25 (+/- 1) **1**
the interquartile range for examination 1 = 31 or follow through **1**

(ii) Examination 2 appears to be the easier one **[1]**
as the median mark is higher **1**
Or as the marks in all the groups is higher

(iii) Complete the grouped frequency table for examination one and estimate the mean and standard deviation of the marks for examination 1.

Length	Cumulative frequency	Frequency
0 - 19	300	300
20 - 39	1100	800
40 - 59	1600	500
60 - 79	1900	300
80-100	2000	100

[2]

Mid-points: 9.5,29.5,49.5,69.5,89.5 (Discrete data) **1**
Mean = (300x 9.5+ 800 x 29.5 + 500 x 49.5 + 300 x 69.5 + 100 x 89.5)/2000 = 81000/2000 **1**
= 40.5 (only 1 mark lost if mid points 10,30 … used) **1**
Var = 4198500/2000 – 40.5^2 = 359 **2**
s.d. = 18.9 **1`**

2 $f(x)=\dfrac{4}{3+x}, \quad x>0 \qquad\qquad g(x)=9-x^2, \quad x\in\mathbb{R}.$

a) $fg(x)$ = f(9-x^2) **[1]**

$= \dfrac{4}{3+9-x^2} = \dfrac{4}{12-x^2},$ **[1]**

($9-x^2$) >0, |x| < 3 **1**

b) (i) $g(x)=5$

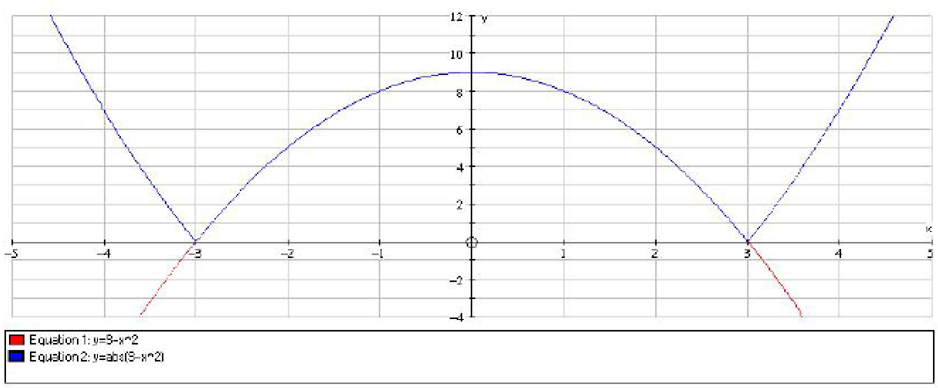
$9-x^2=5$ **[1]**

$x^2=4$

x= ± 2 **[1]**

(ii) function g does not have an inverse as g is a many-one function (or equivalent). **[1]**

c) Sketch the graph of $y=g(x)$. Shape- 1, Max point 1 **[2]**
Using the same axes, sketch the graph $y=|g(x)|$. Centre section 1, sides 1 **[2]**



(i) 0 < f(x) < 4/3 **2**

(ii) $f^{-1}(x)$.: $y=\dfrac{4}{3+x}$

$x=\dfrac{4}{y}-3$ **[1]**

$f^{-1}(x)=\dfrac{4}{x}-3$, 0<x<4/3 **1,1**

3 (a) $\frac{d}{dx}xe^{2x} = e^{2x} + 2xe^{2x}$ with respect to x. 2

(b) $y = (\ln x)^2$ x>0

(i) $\frac{dy}{dx} = 2\ln(x).\frac{1}{x}$ 1

$\frac{d^2y}{dx^2} = \frac{2}{x^2} - \frac{2\ln(x)}{x^2} = \frac{2(1-\ln x)}{x^2}$. 2

(ii) The *gradient* takes its maximum value when $\frac{d^2y}{dx^2} = 0$. 1

$\frac{2(1-\ln x)}{x^2} = 0$ 1

$1 = \ln x$

$x = e$ 1

y = 1 1

(c) (i) $y = \tan^{-1}x$, x = tan y

$\frac{dx}{dy}$ = $\sec^2$y 1

$\sec^2$y = 1 + $\tan^2$y 1

= 1 + x^2

Show that 1

$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sec^2 y} = \frac{1}{1+x^2}$ 1

(ii) $\int_0^1 \frac{1}{1+x^2}dx = \left[\tan^{-1}x\right]_0^1$ 1

$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$ 1

4 (a) $u = \sin x$

(a) Diff w.r.to u: 1 = $\cos x \frac{dx}{du}$ 1

(a) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos 2x \cos x\, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 - 2\sin^2 x)\cos x \frac{dx}{du} du$ M1 A1

$\int_{\frac{1}{2}}^{1} (1 - 2u^2)\, du$ 1

$= \left[u - \frac{2u^3}{3}\right]_{1/2}^{1}$ 1

= (1-2/3) –(1/2- 1/12) = -1/12 1

(b) (i)

$\int_1^2 \frac{1}{\sqrt{(5x-1)}}dx = \int_1^2 (5x-1)^{-1/2}dx$ 1

$= \left[\frac{2(5x-1)^{1/2}}{5}\right]_1^2$ 2

= 6/5 – 4/5 1

= 2/5 1

(ii) Volume = $\int_1^2 \pi y^2 dx$ 1

$= \pi\int_1^2 \frac{1}{5x-1}dx$ 1

$= \left[\frac{\pi}{5}(\ln|5x-1|)\right]_1^2$ 1

$= \frac{\pi}{5}\ln(9/4)$ 1

5 (a) (i) $\dfrac{2}{x(x+1)(x+2)} \equiv \dfrac{A}{x}+\dfrac{B}{(x+1)}+\dfrac{C}{(x+2)}$

$2 \equiv A(x+1)(x+2) + Bx(x+2) + Cx(x+1)$ **1**

$x=0 \Rightarrow$ 2=2A, A=1 **1**

$x=-1 \Rightarrow$ 2 = -B, B=-2 **1**

$x = -2 \Rightarrow$ 2 = 2C, C=1 **1**

$\dfrac{2}{x(x+1)(x+2)} \equiv \dfrac{1}{x}-\dfrac{2}{(x+1)}+\dfrac{1}{(x+2)}$

(b)

(ii) Hence show that

$$\int_2^4 \frac{2}{x(x+1)(x+2)}\,dx = 2\ln 3 - 2\ln 5 .$$

$\int_2^4 \dfrac{2}{x(x+1)(x+2)}\,dx = \int_2^4 \dfrac{1}{x}-\dfrac{2}{(x+1)}+\dfrac{1}{(x+2)}\,dx$ **1**

$= \left[\ln|x| - 2\ln|x+1| + \ln|x+2|\right]_2^4$ **1**

$= \ln 4 - 2\ln 5 + \ln 6 \quad -(\ln 2 - 2\ln 3 + \ln 4)$ **1**

= ln 4 – 2ln 5 + ln2 + ln3 - ln2 +2ln3 - ln4

= 3 ln 3 – 2 ln 5 q.e.d. **1**

(a) (i) $\dfrac{d}{dx}\left(\dfrac{\sin x}{\cos x}\right) = \dfrac{\cos^2 x - (-\sin^2 x)}{\cos^2 x}$ **1**

$= \dfrac{1}{\cos^2 x}$ **1**

$= \sec^2 x$ **1**

(ii) $\int x \sec^2 x\,dx$

Let u = x and dv/dx = $\sec^2$ x, **1**

v=tan x **1**

I = x tanx - $\int \tan x\,dx$ **1**

I = x tan x + ln cos x + C **1**

6 (a)

(i) $\sin(x+\alpha) = 2\sin(x-\alpha)$

$\sin x\cos\alpha + \cos x\sin\alpha = 2\sin x\cos\alpha - 2\cos x\sin\alpha$ **1**

$3\cos x\sin\alpha = \sin x\cos\alpha$ (divide by $3\cos x\cos\alpha$) **1**

$\tan x = 3\tan\alpha$ **1**

$\sin\left(2y+\dfrac{\pi}{4}\right) = 2\sin\left(2y-\dfrac{\pi}{4}\right)$ for $0 < y < \pi$

$\tan 2y = 3\tan(\frac{\pi}{4})$ **1**

tan 2y = 3, y = 1.249 or π +1.249 **1**

y = 0.625, 2.20 (2.195) **1**

b (i) $a = \pi/2$ **1**

(ii) $\tan^{-1} x = \pi/4$. **1**

x= $\tan(\pi/4)$ = 1 **1**

(iii) Translation, 1unit along the x axis $\begin{pmatrix}1\\0\end{pmatrix}$ **2**

Stretch, s.f. 2, in y direction (parallel to the y axis) **2**

(iv) x=1.54: y= $\tan^{-1}1.54$ = 0.99

$y = 2\tan^{-1}(x-1)$ = 2* $\tan^{-1}$ 0.54 = 2* 0.495 = 0.99 **1**

7

a) $\int \frac{1}{20-x}dx = \int \frac{1}{10}dt$ **1**

$-\ln(20-x) = \frac{1}{10}t + c$ x=5 when t=0 $\Rightarrow$ c=$-\ln 15$ **1**

$-\ln(20-x) = \frac{1}{10}t - \ln 15$ **1**

$-\frac{1}{10}t = \ln\frac{(20-x)}{15}$ **1**

$x = 20 - 15e^{-\frac{t}{10}}$ **1**

When x=10, t/10 = ln(15/10) **1**

= 4.05 3 s.f. **1**

b

(i) $\frac{dx}{dt}$ = -k$\sqrt{x}$ **1**

$\int \frac{1}{\sqrt{x}}dx = \int -k\,dt$ **1**

$2\sqrt{x} = -kt + c$ **1**

(ii) x=400cm when t=0; **c=40** **1**

x=200, $\frac{dx}{dt}$=-1

-1=-k10$\sqrt{2}$ **1**

k = 0.0707 **1**

(iii) $2\sqrt{x} = -0.0707t + 40$

1,1

8 a) The position vectors of 3 points A, B, C are

a = 4**i** + 2**j** - **k** , **b** = -2**i** +26**j** + 11**k** , **c** = 16**i** + 17**j** + 2**k**

Show that vector 2**i** - 3j + 7k is perpendicular to $\overrightarrow{AB}$ and also perpendicular to $\overrightarrow{AC}$.

$\overrightarrow{AB}$ = **b – a** **M1**

= -2**i** +26**j** + 11**k** – **(**4**i** + 2**j** - **k)** **A1**

= -6**i** +24**j +** 12**k**

$\overrightarrow{AC}$ = **c – a**

= 16**i** + 17**j** + 2**k** – **(**4**i** + 2**j** - **k)**

= 12**i** +15**j** + 3**k** **1**

(2**i** - 3j + 7k) **.** (-6**i** +24**j +** 12**k) =** -12 -72 + 84 = 0

(2**i** - 3j + 7k) **.** (12**i** +15**j** + 3 **k)** = 24 -45 +21 = 0 **1**

$\therefore$ vector 2**i** - 3j + 7k is perpendicular to $\overrightarrow{AB}$ and also perpendicular to $\overrightarrow{AC}$. **1**

b)

(i) The coordinates of the point of intersection.

Equating the x and y components

5 + λ = 2 - 3 μ **1**

3 – 2 λ = -11 -4 μ **1**

μ = -2, λ = 3 **2**

Point of intersection: (8, -3, 4) **1**

(ii) The value of the constant a , **1**

a – 10 = -2 +6 **1**

a = 14

(iii) The acute angle between the lines.

(**i** – 2**j** + 2**k**). (-3**i** – 4**j** + 5**k**) = 3 x $\sqrt{50}$ $\cos\theta$. **M1**

$\cos\theta$ = 15/(15 $\sqrt{2}$ = 1/$\sqrt{2}$ **A1**

θ = 45^0 or $\pi/4$ **1**