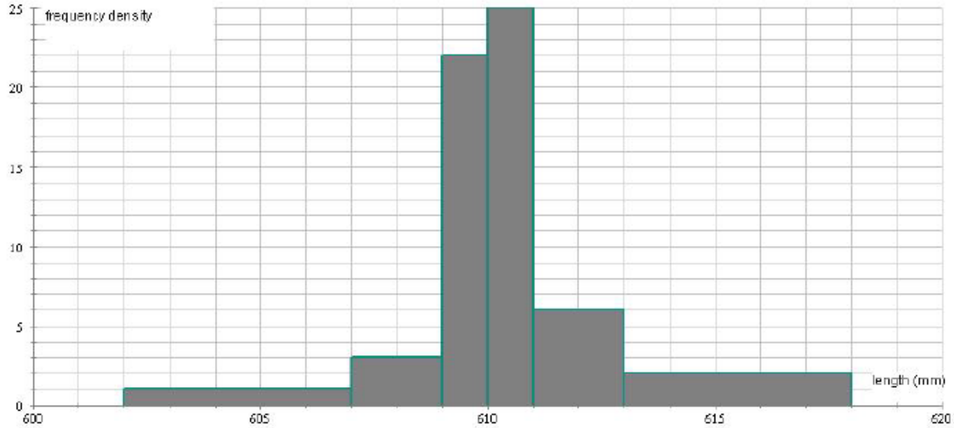
Practice Exam 2 (Engineers)

1 (a) A man makes garden fence posts which should be 610 mm long.
The lengths of 80 posts are measured.
Their lengths are illustrated in the histogram below.



(i) Copy and complete the grouped frequency table below, stating the number of posts in each of the classes in the histogram. **[3]**

Length	Frequency
602 to 607	
….	

(ii) Estimate the mean length of the posts. Give your answer correct to 1 decimal place. **[3]**

(iii) Estimate the standard deviation of the lengths of the posts. **[4]**

(b) Two hundred and fifty policemen have the following heights.

Height (cm)	No. of men
$160 \le$ height<165	18
$165 \le$ height<170	37
$170 \le$ height<175	60
$175 \le$ height<180	65
$180 \le$ height<185	48
$185 \le$ height<190	22

Find the cumulative frequencies and plot the cumulative curve. **[3]**
Use the curve to estimate:
(i) the median height; **[1]**
(ii) the lower quartile height. **[1]**

2 (a) The function f is defined by

$$f(x) = x^2 + 3.$$

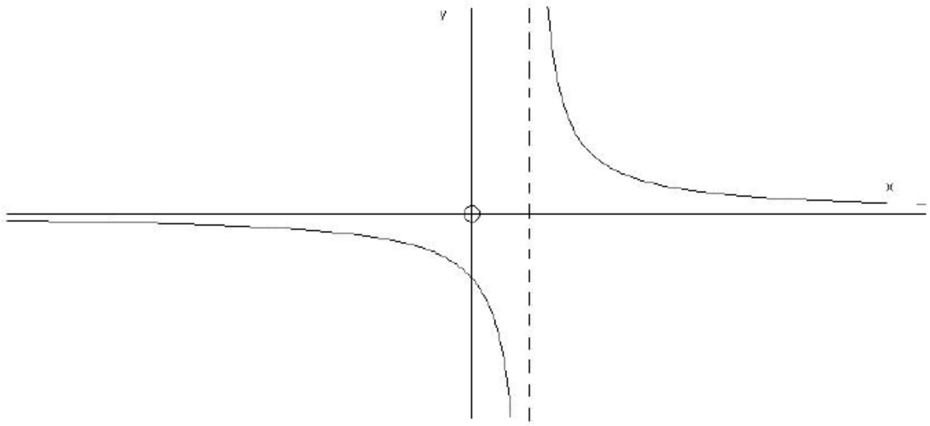
(i) Write down the range of $f(x)$. **[1]**

(ii) Explain why $f(x)$ has no inverse. **[2]**
Suggest a suitable domain so that $f^{-1}(x)$ does exist.

(b) The function g is defined by

$$g(x) = \frac{5}{x-2}, \qquad x \neq 2.$$

The diagram shows a sketch of $y = g(x)$.



(i) Sketch the curve $y = g^{-1}(x)$. Write the equations of the asymptotes. **[5]**

(ii) Calculate the exact x coordinates of the points for which $g(x) = x$. **[3]**

(iii) Find the composite function $gf(x)$, where $f(x)$ is the function in (a) above, and determine its range. **[4]**

3 (a) Differentiate $e^{-x}\sin 2x$ with respect to x. **[2]**

(b) Given that $y = \tan x + 2\cos x$, find the exact value of $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$. **[4]**

(c) Find the gradient of the curve **[5]**

$$y = \frac{1}{(4x+3)^2}$$

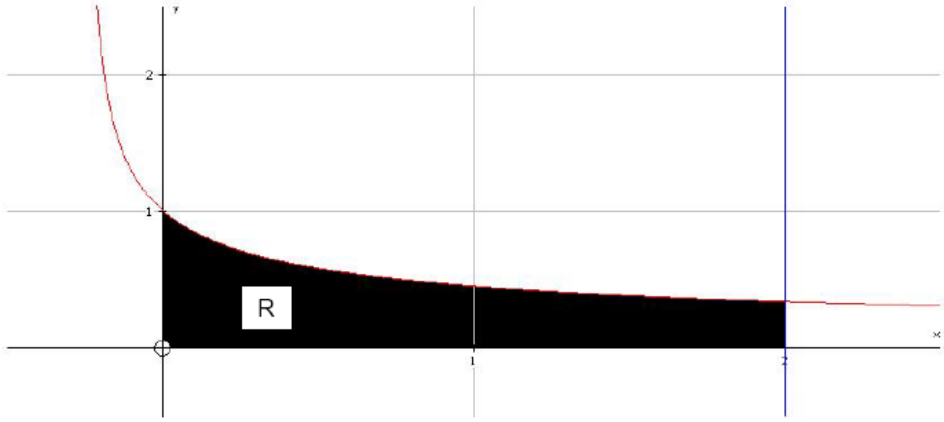
at the point where $x = 1/4$.

(d) Given that $x = \tan\frac{1}{2}y$, prove that $\frac{dy}{dx} = \frac{2}{1+x^2}$. **[4]**

4 (a) Use the substitution $u = x - 1$ to evaluate **[7]**

$$\int_2^5 \frac{x}{\sqrt{(x-1)}}\,dx\ .$$

(b)



The diagram shows the curve

$$y = \frac{1}{\sqrt{(4x+1)}}\ .$$

The shaded region, R, is enclosed by the curve, the axes and the line $x = 2$.

(i) Show that the exact value of the area of the region R is 1. **[4]**

(ii) The region R is rotated completely about the x axis. Find the exact value of the volume of the solid formed. **[4]**

5 (a) Let $\mathrm{I} = \int \frac{1}{x(1+\sqrt{x})^2}dx$

(i) Show that the substitution $u = \sqrt{x}$ transforms I to $\int \frac{2}{u(1+u)^2}du$. **[3]**

(ii) Express $\frac{2}{u(1+u)^2}$ in the form $\frac{A}{u} + \frac{B}{(1+u)} + \frac{C}{(1+u)^2}$ **[4]**

(iii) Use your result from (ii) to find I . **[3]**

(b) Find $\int_0^1 x e^{-2x}\,dx$, giving your answer in terms of e. **[5]**

6 (a) .

(i) Write down the formula for $\tan 2x$ in terms of $\tan x$. **[1]**

(ii) By letting $\tan x = t$, show that the equation **[3]**

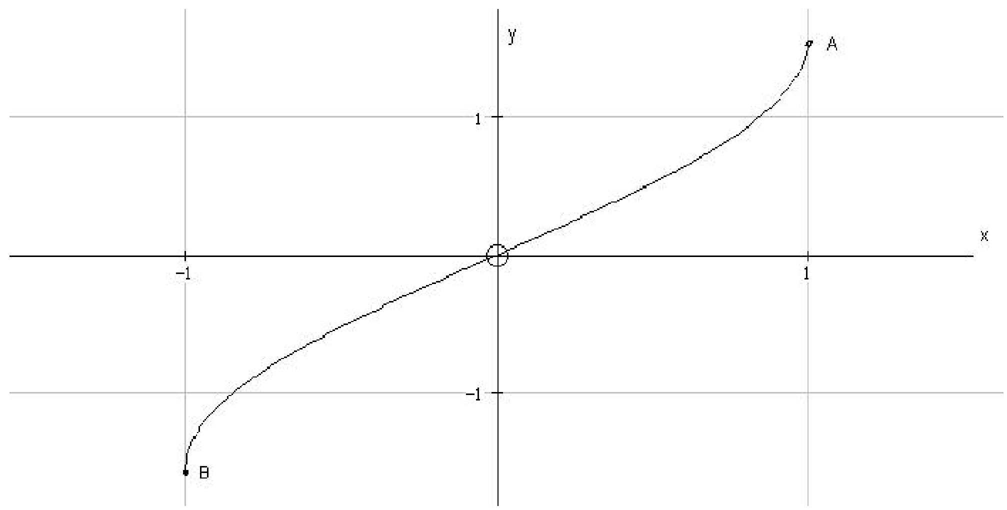
$$4\tan 2x + 3\cot x \sec^2 x = 0$$

becomes $$3t^4 - 8t^2 - 3 = 0$$

(iii) Hence find all the solutions of the equation **[5]**

$$4\tan 2x + 3\cot x \sec^2 x = 0 \qquad \text{for } 0 \le x \le 2\pi.$$

(b)



The diagram shows the curve $y = \sin^{-1} x$ and its end points A and B.

(i) State the coordinates of A and B. **[2]**

(ii) Find the value of x for which $\sin^{-1} x = \pi/6$ **[1]**

(iii) The equation of another curve is $y = 2\sin^{-1}(x-1)$. **[3]**
Describe the geometrical transformations by which this curve can be obtained from the curve with equation $y = \sin^{-1} x$.

7

(a) Solve the equation $\dfrac{dy}{dx} = y\cos x$, given that $y = 1$ when $x = \dfrac{\pi}{2}$. **[5]**
Express your answer in the form $y = f(x)$.

(b) The population of England, P, is increasing yearly at a rate proportional to the population.

(i) Form a differential equation to show this. **[2]**

(ii) Solve the equation, giving P as a function of t, time in years, given $P=P_0$ when t=0. **[5]**

(iii) Given that the population at the beginning of 1980 was 50 million and at the beginning of 2000 was 60 million, find the values of the constants in your equation. **[3]**

8 Referred to a fixed origin O, the points A and B have position vectors $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $-\mathbf{i} + \mathbf{j} + 9\mathbf{k}$ respectively.

(i) Show that *OA* is perpendicular to *AB*. **[5]**

(ii) Find, in vector form, the equation of the line L_1 which passes through A and B. **[2]**

(iii) The line L_2 has equation $\mathbf{r} = (8\mathbf{i} + \mathbf{j} - 6\mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$. Show that the lines L_1 and L_2 intersect and find their point of intersection. **[5]**

(iv) Calculate, to the nearest tenth of a degree, the acute angle between L_1 and L_2. **[3]**

Practice Exam 2 Answers

1 a)

Lengths	Frequency
602 to 607	5
607 to 609	6
609 to 610	22
610 – 611	25
611-613	12
613-618	10

[2] freq

[1] total 80

F	x (mid-point)	fx	fx^2
5	604.5	3022.5	1827101
6	608	3648	2217984
22	609.5	13409	8172786
25	610.5	15262.5	9317756
12	612	7344	4494528
10	615.5	6155	3788403
Totals			
80		48841	29818558

[1] x

[1] fx

[1] mean

[1] fx^2

Mean = 48841/80 = 610.5 (1d.p.)

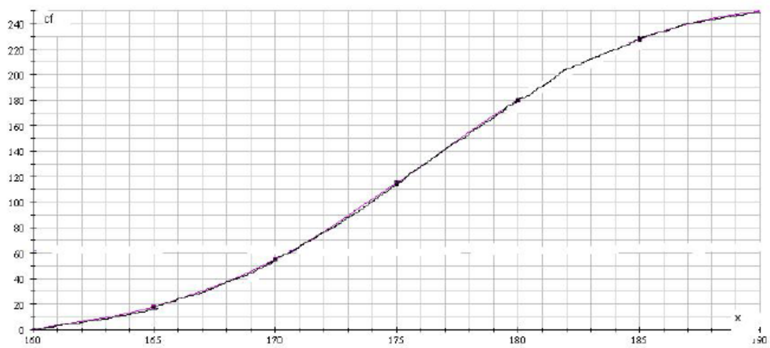
Var = 29818558/80 - 610.5^2 = 6.456 **[1] var**

Standard Deviation = 2.54 (3 s.f.) **[1] sd**

(b) Plot (160,0), (165,18), (170,55), (175,115), (180.180), (185,228), (190,250) **[1] uci**

[1] axes

[1] plot



[1] median

Median (cf = 125): 175.8 cm (+/- 0.4)

Lower quartile (c,f, = 62.5): 170.6 cm (+/- 0.4) **[1] LQ**

2 (i) $f(x) \geq 3$ **[1]**

(ii) $f(x)$ has no inverse as it is a many-one function
or as $f(x)$ is not a one-one function…or equivalent **[1]**
Suggest a suitable domain so that $f^{-1}(x)$ does exist: $x \geq 0$ **[1]**

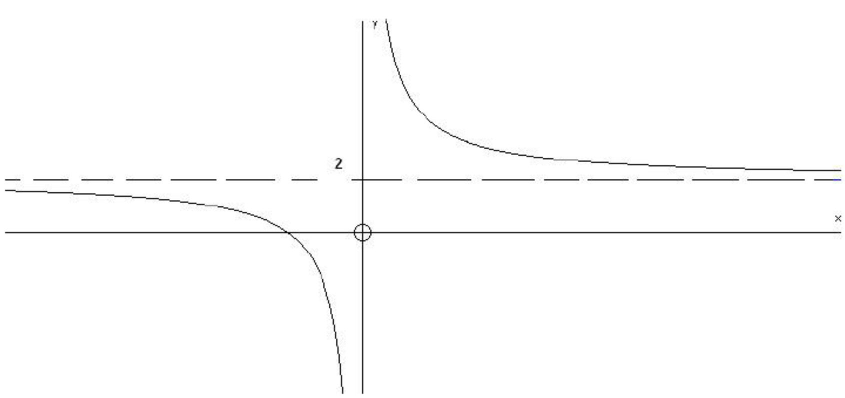
(b)

Sketch the curve $y = g^{-1}(x)$.

Two branches drawn **[1]**

Both branches drawn to the asymptotes **[2]**

Write the equations of the asymptotes: y=2, x=0 **[2]**



ii) Calculate the exact x coordinates of the points for which $g(x) = x$.

$$x = \frac{5}{x-2}$$

$x^2 - 2x - 5 = 0$ [1]

x = (2 +/- (4+20)^0.5)/2 [1]

x = 1 +/- √6 [1]

iii) Find the composite function $gf(x)$ and determine its range.

$gf(x) = g(x^2 + 3)$ [1]

$$= \frac{5}{x^2+3-2}$$

$$= \frac{5}{x^2+1}$$ [1]

Max value when x=0, $gf(x) = 5$ [1]

Min value when x $\to \infty$, $gf(x) \to 0$ [1]

Range: $0 < gf(x) \le 5$

3 (a) Differentiate $e^{-x}\sin 2x$ with respect to x.

$$\frac{d}{dx}(e^{-x}\sin 2x) = -e^{-x}\sin 2x + e^{-x}2\cos 2x$$

2

(b) $y = \tan x + 2\cos x$,

$$\frac{dy}{dx} = \sec^2 x - 2\sin x$$

2

$$At\ x = \frac{\pi}{4},\ \frac{dy}{dx} = 2 - \sqrt{2}$$

2

(c) $y = \frac{1}{(4x+3)^2}$ = $(4x + 3)^{-2}$ 1

$\frac{dy}{dx} = -2(4x+3)^{-3}4$ or substitute u = (4x+3) 2

$$\frac{dy}{dx} = -8(4x+3)^{-3}$$

when $x = 1/4$, $\frac{dy}{dx} = -8(4)^{-3}$ 1

$\frac{dy}{dx} = -\frac{1}{8}$ (or 0.125)

1

(d) $x = \tan\frac{1}{2}y$

$$\frac{dx}{dy} = \frac{1}{2}\sec^2\left(\frac{y}{2}\right)$$ 1

$$\frac{dy}{dx} = \frac{2}{\sec^2(\frac{1}{2}y)}$$ 1

$$= \frac{2}{1+\tan^2(\frac{1}{2}y)}$$ 1

$$= \frac{2}{1+x^2}$$ 1

4 (a) $u = x - 1$

$\frac{du}{dx} = 1$ 1

At x=2, u=1 and at x=5, u=4 1

$$\int_2^5 \frac{x}{\sqrt{(x-1)}}dx = \int_1^4 \frac{u+1}{u^{\frac{1}{2}}}\left(\frac{dx}{du}\right)du$$ 1

$$= \int_1^4 (u^{\frac{1}{2}} + u^{-\frac{1}{2}})du$$ 1

$$= \left[\frac{2u^{\frac{3}{2}}}{3} + \frac{2u^{\frac{1}{2}}}{1}\right]_1^4$$ 1

$$= 6\tfrac{2}{3}$$ 2

(b) (i) Area = $\int_0^2 \frac{1}{\sqrt{(4x+1)}}dx$ **1**

$= \int_0^2 (4x+1)^{-1/2}dx$ **1**

$= \left[\frac{2}{4}(4x+1)^{1/2}\right]_0^2$ **1**

$= \frac{3}{2} - \frac{1}{2}$

= 1 **1**

(ii) Volume = $\int_0^2 \pi y^2 dx$ **1**

$= \pi\int_0^2 \frac{1}{4x+1}dx$ **1**

$= \left[\frac{\pi}{4}(\ln|4x+1|)\right]_0^2$ **1**

$= \frac{\pi}{4}\ln 9$ **1**

5

(a) $I = \int \frac{1}{x(1+\sqrt{x})^2}dx$

1

(i) $u = \sqrt{x}$, $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$, $\frac{dx}{du} = 2\sqrt{x}$ or $x = u^2$, $\frac{dx}{du} = 2u$

M1 A1

$I = \int \frac{1}{x(1+\sqrt{x})^2}\frac{dx}{du}du = \int \frac{1}{u^2(1+u)^2} 2u\ du$

$= \int \frac{2}{u(1+u)^2}du$

(ii) $\frac{2}{u(1+u)^2} = \frac{A}{u} + \frac{B}{(1+u)} + \frac{C}{(1+u)^2}$

2 = A(1+u)^2 + Bu(1+u) + Cu **1**

u=0 ⇒ 2 = A

u = -1 ⇒ 2 = -C, C = -2

u = 1 ⇒ 2 = 8 + 2B -2, B = -2 **3**

$\frac{2}{u(1+u)^2} = \frac{2}{u} - \frac{2}{(1+u)} - \frac{2}{(1+u)^2}$

(iii) $I = 2\int\left(\frac{1}{u} - \frac{1}{(1+u)} - \frac{1}{(1+u)^2}\right)du$

= 2(ln u – ln|1+u| + 1/(1+u)) **2**

$= 2\ln\left(\frac{\sqrt{x}}{1+\sqrt{x}}\right) + \frac{2}{1+\sqrt{x}}$ + C **1**

(b) Find $\int_0^1 xe^{-2x}dx$, giving your answer in terms of e.

Let u = x and dv/dx = e^{-2x} **1**

v = $\frac{e^{-2x}}{-2}$ **1**

$\int_0^1 xe^{-2x}dx = \left[x\frac{e^{-2x}}{-2}\right]_0^1 - \int_0^1 \frac{e^{-2x}}{-2}dx$ **1**

$= \left[\frac{-xe^{-2x}}{2} - \frac{e^{-2x}}{4}\right]_0^1$ **1**

$= -\frac{e^{-2}}{2} - \frac{e^{-2}}{4} - (0 - \frac{1}{4})$

$= \frac{1}{4} - \frac{3e^{-2}}{4}$ **1**

6

(a) (i) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$. **1**

(ii) $4\left(\dfrac{2t}{1-t^2}\right)+3\dfrac{1}{t}(1+t^2)=0$ **1**

1

$8t^2+3(1-t^4)=0$ **1**

$3t^4-8t^2-3=0$

$(3t^2+1)(t^2-3)=0$ **1**

$t^2 \neq -1/3,\ t=\pm\sqrt{3}$ **1**

tan x = $\sqrt{3}$, x = $\pi/3,\ 4\pi/3$ **2**

tan x = -$\sqrt{3}$, x = $2\pi/3,\ 5\pi/3$ **1**

(b) (i) A: (1, $\pi/2$), B: (-1, $-\pi/2$) **1+1**

(ii) $\sin^{-1}$ x = $\pi/6$, x=sin($\pi/6$) = 1/2 **1**

$y=2\sin^{-1}(x-1)$.

translation $\begin{pmatrix}1\\0\end{pmatrix}$, **1**

stretch, scale factor 2, parallel to the y axis **2**

7

a) $\int \frac{1}{y}\,dy = \int \cos x\,dx$ Separating the variables **1**

$\ln y = \sin x + c$ **1**

$y=1$ when $x=\dfrac{\pi}{2} \Rightarrow c=-1$ **1**

$\ln y = \sin x - 1$ **1**

$y=e^{\sin x-1}$ **1**

b) (i)

$\dfrac{dP}{dt}=kP$ **2**

(ii) $\int \frac{1}{P}\,dP = \int k\,dt$ **1**

$\ln P = kt + c$ **1**

P=P_o when t=0 $\Rightarrow c=\ln P_o$ **1**

$\ln\dfrac{P}{P_o}=kt$ **1**

$P=P_o e^{kt}$ **1**

(iii) t=0, P= P_o = 50,000,000

t=20, P=60000000 **1**

k=1/20(ln 6/5)

= 0.00912 (3 s.f.) **2**

8 (i) Show that *OA* is perpendicular to *AB*.

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$ **M1**

= -**i** + **j** + 9**k** - (3**i** – **j** + 2**k**)

= -4**i** + 2**j** + 7**k** **A1**

$\overrightarrow{OA}.\overrightarrow{AB}$ = (3**i** – **j** + 2**k**) . (-4**i** + 2**j** + 7**k**) **M1**

= -12 -2 +14 = 0 **A1**

$\therefore$ **OA** is perpendicular to **AB,** **1**

(ii) Find in vector form, an equation of the line L_1 which passes through A and B. **M1**

r = 3**i** – **j** + 2**k** + λ (-4**i** + 2**j** + 7**k**) is an equation of the line L_1. **A1**

(iii) The line L_2 has equation r = (8**i** + **j** – 6**k**) + μ (**i** – 2**j** – 2**k**). Show that the lines L_1 and L_2 intersect and find their point of intersection.

Equating the x and y components of L_1 *and* L_2;

3 - 4λ = 8 + μ **1**

-1 +2λ = 1 -2 μ **1**

λ = -2, μ = 3 **1**

Check for consistency on the z component: L_1: z component is 2 -14 = -12

L_2: z component is -6 -6 = -12

The point of intersection: 8**i** -5**j** -12**k** **1**

1

(iv) Calculate, to the nearest tenth of a degree, the acute angle between L_1 and L_2.

(-4**i** + 2**j** + 7**k**) . (**i** – 2**j** – 2**k**) = $\sqrt{69}\sqrt{9}\cos\theta$ **M1**

-4 -4 -14 = $\sqrt{69}\sqrt{9}\cos\theta$ **A1**

$\cos\theta$ = 28.0^0 to nearest 0.1 degree **1**