Practice Exam 1 (Engineers)

Section A. Answer ALL questions in this section. They are not equally weighted.

Question A1

Differentiate $x^2 \ln x^2$ with respect to x. **[3]**

Question A2

Find the inverse of $f(x) = \dfrac{2x-3}{x+5}$, $x \in \mathbb{R}$, $x \neq -5$. **[4]**

Question A3

Prove the identity $\operatorname{cosec}\theta - \sin\theta = \cot\theta\cos\theta$. **[4]**

Question A4

Find all the solutions between $0°$ and $360°$ to the equation $\sin 2\theta = -0.6$. **[4]**

Question A5

If $\mathbf{a} = \begin{pmatrix} -5 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 0 \\ -4 \end{pmatrix}$, find $|3\mathbf{a} + 2\mathbf{b}|$. **[4]**

Question A6

Find the equation of the tangent to the curve $y = \dfrac{1}{(2x-3)^5}$ **[5]**

Question A7

Use integration by parts to find $\int_0^1 x^2 e^x \, dx$,

giving your answer in terms of e. **[5]**

Question A8

Solve the differential equation

$$y\frac{dy}{dx} = 4\sin x$$

given that $y = 1$ when $x = 0$. Express your answer in the form $y = f(x)$. **[5]**

Question A9

Use the substitution $u = 3x - 1$ to evaluate $\int_{1/3}^{2/3} \dfrac{3}{9x^2 - 6x + 2} dx$. **[6]**

Section B. Answer FOUR questions from this section. The questions are equally weighted

Question B1

(a) Find the stationary value (turning point) of $y = e^{-x}\sin x$ for $0 < x < \pi$,

and decide if this is a maximum or a minimum. **[7]**

(b) Solve $4\sin 2\theta + 3 = 4\cos\theta + 6\sin\theta$ for $0 < \theta < 2\pi$. **[8]**

Question B2

The functions f and g are defined as follows:

$$f(x) = \tan\left(\frac{\pi x}{2}\right) \qquad -1 < x < 1$$

$$g(x) = |x| \qquad x \in \mathbb{R}$$

(i) Find the range of $f(x)$. **[1]**

(ii) Write down $gf(x)$ and find the value of $g(f(-1/3))$. **[3]**

(iii) Sketch the graphs of $f(x)$ and $gf(x)$ using the same axes. **[4]**

(iv) Find an expression for $f^{-1}(x)$.

Explain why $g(x)$ has no inverse. **[4]**

(v) One of f and g is an odd function.

State which one is odd, giving your reason.

Explain why the other function is not odd. **[3]**

Question B3

(a) Let $f(x) = 1 - \dfrac{1}{x^2}$.

(i) Sketch the curve $y = f(x)$ for $1 \le x \le 2$. **[1]**

(ii) The area is rotated completely about the x-axis.

Find the exact value of the volume of the solid formed. **[4]**

(b) (i) Express $\dfrac{1-x}{(x+1)(x^2+1)}$ in the form $\dfrac{A}{x+1} + \dfrac{B}{x^2+1} + \dfrac{Cx}{x^2+1}$. **[4]**

(ii) Use your result from (i) to show that

$$\int_0^{3/4} \frac{1-x}{(x+1)(x^2+1)}\,dx = \ln\left(\frac{7}{5}\right). \qquad \textbf{[6]}$$

Question B4

Let A, B and C be points with position vectors

$$\mathbf{a} = \mathbf{i} - 5\mathbf{j} + 3\mathbf{k},\ \mathbf{b} = 3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k},\ \mathbf{c} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k},$$

respectively, with respect to the origin O.

(i) Show that $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$. **[3]**

(ii) Let L_1 be the line passing through C parallel to $\overrightarrow{OA}$.

Find the vector equation of L_1. **[2]**

(iii) Show that L_1 intersects L_2,

where $L_2 = 3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k} + \mu(-\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$,

and find the point of intersection. **[6]**

(iv) Find the acute angle between the lines L_1 and L_2. **[4]**

Question B5

(a) (i) Show that the equation $e^x = 2x + 1$ has a root between 1.2 and 1.3. **[3]**

(ii) Starting with initial value $x_0 = 1.2$,

use the Newton-Raphson method, twice,

to give a better approximation to the root of the equation in (i)

giving your final answer correct to 3 decimal places. **[5]**

(b) In a chemical reaction a substance A is transformed into a substance B.

Initially there are a molecules of A and no molecules of B.

After time t x molecules of A are left.

The rate of conversion is proportional to the amount of A that remains.

(i) Write down the differential equation which models this reaction. **[2]**

(ii) Find the general solution of the equation. **[5]**

Question B6

The grouped frequency table below shows the number of laptops sold per day in a computer store during a period of 200 working days.

Number of laptops sold	Frequency
46 – 60	20
61 – 65	30
66 – 70	55
71 – 75	50
76 – 80	35
81 – 95	10

(i) Draw a histogram to illustrate this set of data. **[6]**

(ii) Calculate the mean number of laptops sold per day. **[4]**

(iii) Find the standard deviation of the data. **[5]**

Practice Exam 1 Answers

Section A

A1 $\frac{dy}{dx} = 2x\ln x^2 + x^2 \frac{1}{x^2}2x$ **[2]**

$= 2x\ln x^2 + 2x$ **[1]**

A2 Let $y = \frac{2x-3}{x+5}$

$yx + 5y = 2x - 3$ **[1]**

$yx - 2x = -5y - 3$ **[1]**

$x = \frac{-5y-3}{y-2}$ **[1]**

$f^{-1}(x) = \frac{-5x-3}{x-2}, \ \ x \neq 2$ **[1]**

A3 $\operatorname{cosec}\theta - \sin\theta = \frac{1}{\sin\theta} - \sin\theta$ **[1]**

$= \frac{1-\sin^2\theta}{\sin\theta}$ **[1]**

$= \frac{\cos^2\theta}{\sin\theta}$ **[1]**

$= \frac{\cos\theta}{\sin\theta}\cos\theta$

$= \cot\theta\cos\theta$ **[1]**

A4 $\sin^{-1} 0.6 = 36.87°$ from a calculator **[1]**

The sine function takes negative values in the third and fourth quadrants

So $2\theta = 180° + 36.87°$ or $360° - 36.87°$ or $360° +$ either of these. **[1]**

So the possible values are $108.4°, 161.6°, 288.4°, 341.6°$. **[2]**

A5 $3\mathbf{a} + 2\mathbf{b} = 3\begin{pmatrix} -5 \\ 2 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 0 \\ -4 \end{pmatrix}$

$= \begin{pmatrix} -15 \\ 6 \\ 9 \end{pmatrix} + \begin{pmatrix} 8 \\ 0 \\ -8 \end{pmatrix}$ **[1]**

$= \begin{pmatrix} -7 \\ 6 \\ 1 \end{pmatrix}$ **[1]**

So $|3\mathbf{a} + 2\mathbf{b}| = \sqrt{49+36+1} = \sqrt{86} \approx 9.27$ **[2]**

A6 $\frac{dy}{dx} = \frac{-5}{(2x-3)^6} 2 = \frac{-10}{(2x-3)^6}$ **[1]**

$= -10$ at the point $(2,1)$. **[1]**

The equation of the tangent is $y = -10x + c$. **[1]**

$1 = -10 \times 2 + c$, $c = 21$. **[1]**

$y = -10x + 21$ **[1]**

A7 $\int_0^1 x^2 e^x \, dx = \left[x^2 e^x\right]_0^1 - \int_0^1 2x e^x \, dx$ **[1]**

$= \left[x^2 e^x - 2x e^x\right]_0^1 + \int_0^1 2 e^x \, dx$ **[1]**

$= \left[x^2 e^x - 2x e^x + 2 e^x\right]_0^1$ **[1]**

$= e - 2e + 2e - 0 + 0 - 2$ **[1]**

$= e - 2.$ **[1]**

A8 $\int y \, dy = \int 4 \sin x \, dx$ **[1]**

$\frac{y^2}{2} = -4\cos x + c$ **[2]**

$\frac{1}{2} = -4 + c$

$c = \frac{9}{2}$ **[1]**

$\frac{y^2}{2} = -4\cos x + \frac{9}{2}$

$y = \sqrt{9 - 8\cos x}$ **[1]**

A9 $u = 3x - 1$, $x = \frac{u+1}{3}$, $dx = \frac{du}{3}$. **[2]**

$\int_{1/3}^{2/3} \frac{3}{9x^2 - 6x + 2} \, dx = \int_0^1 \frac{1}{u^2 + 1} \, du$ **[2]**

$= \left[\tan^{-1} u\right]_0^1$ **[1]**

$= \frac{\pi}{4} (\approx 0.785).$ **[1]**

Section B

B1(a) $\dfrac{dy}{dx} = -e^{-x}\sin x + e^{-x}\cos x$ **[1]**

$= 0$ at a stationary point. **[1]**

Then $\sin x = \cos x$, $\tan x = 1$, $x = \dfrac{\pi}{4}(\approx 0.785)$. **[1]**

Here $y = e^{-\pi/4}\dfrac{1}{\sqrt{2}}$. (Accept 0.322 or 0.323.) **[1]**

$$\frac{d^2y}{dx^2} = e^{-x}\sin x - e^{-x}\cos x - e^{-x}\cos x - e^{-x}\sin x = -2e^{-x}\cos x.$$

When $x = \pi/4$, $\dfrac{d^2y}{dx^2} = -2e^{-\pi/4}\cos(\pi/4) = -\sqrt{2}\,e^{-\pi/4} < 0$. **[2]**

So the stationary value is a maximum. **[1]**

(b) $8\sin\theta\cos\theta + 3 - 4\cos\theta - 6\sin\theta = 0$ **[2]**

$(2\sin\theta - 1)(4\cos\theta - 3) = 0$ **[2]**

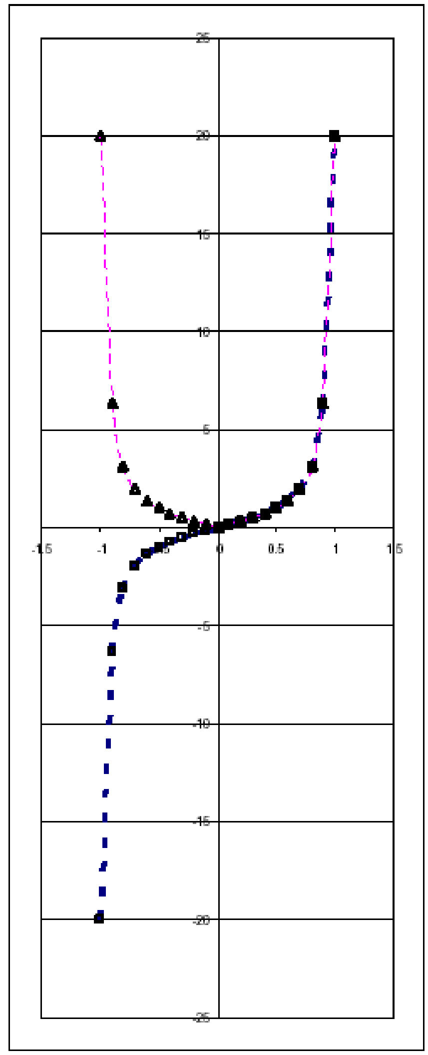
$\sin\theta = \dfrac{1}{2}$ or $\cos\theta = \dfrac{3}{4}$. **[2]**

$\theta = 0.524, 0.723, 2.62, 5.56$ **[2]**

B2(i) The range is $\mathbb{R}$ or $-\infty < x < \infty$. **[1]**

(ii) $f(-1/3) = \tan\left(-\dfrac{\pi}{6}\right) = -\dfrac{1}{\sqrt{3}}(\approx -0.577)$, $g(f(-1/3)) = \dfrac{1}{\sqrt{3}}(\approx 0.577)$. **[1]**

(iii)



Shape of $f(x)$ **[1]**

Passes through origin **[1]**

tends to $\pm\infty$ at ± 1 **[1]**

shape of $g(f(x))$ **[1]**

(iv) $f^{-1}(x) = \dfrac{2}{\pi}\tan^{-1} x$ **[2]**

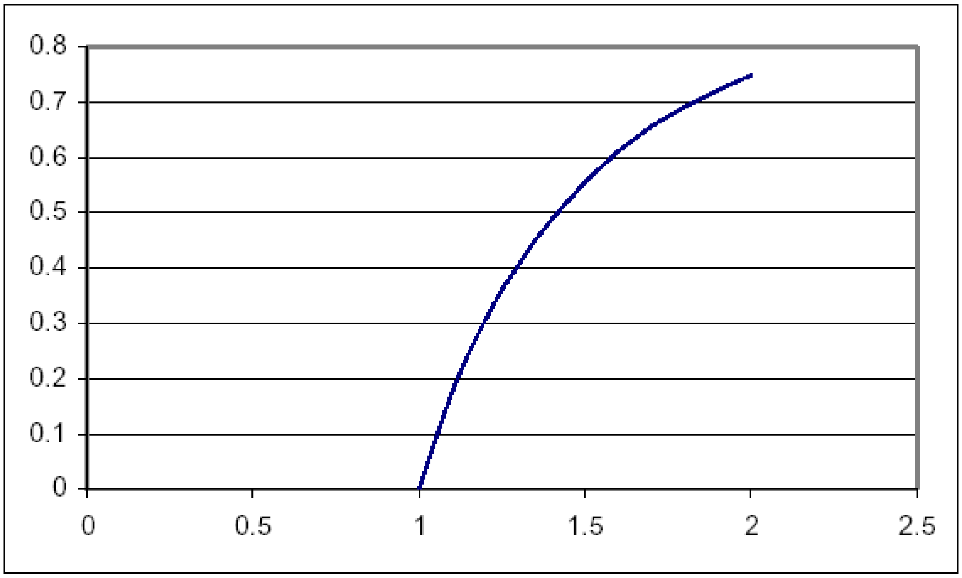
$g(x)$ has no inverse because it is not one-one. $g(-x) = g(x)$ for all x. **[2]**

(v) f is the odd function **[1]**

since $f(-x) = -f(x)$ for all x. **[1]**

g is not odd since $g(-x) = g(x) \neq -g(x)$ except when $x = 0$. **[1]**

B3(a)(i)



Graph **[1]**

(ii) $\pi\int_1^2\left(1-\frac{1}{x^2}\right)^2 \mathrm{d}x = \pi\int_1^2\left(1-\frac{2}{x^2}+\frac{1}{x^4}\right)\mathrm{d}x$ **[1]**

$= \pi\left[x+\frac{2}{x}-\frac{1}{3x^3}\right]_1^2$ **[1]**

$= \pi\left[2+1-\frac{1}{24}-1-2+\frac{1}{3}\right]$ **[1]**

$= \frac{7\pi}{24}$. **[1]**

(b)(i) $\frac{A}{x+1}+\frac{B}{x^2+1}+\frac{Cx}{x^2+1} = \frac{A(x^2+1)+B(x+1)+Cx(x+1)}{(x+1)(x^2+1)}$. **[1]**

This equals $\frac{1-x}{(x+1)(x^2+1)}$

$x=-1 \Rightarrow 2A=2 \Rightarrow A=1$. **[1]**

$x=0 \Rightarrow A+B=1 \Rightarrow B=0$. **[1]**

Equating coefficients of $x^2 \Rightarrow A+C=0 \Rightarrow C=-1$. **[1]**

Given expression equals $\frac{1}{x+1}-\frac{x}{x^2+1}$.

(ii) Given integral equals $\int_0^{3/4}\left(\frac{1}{x+1}-\frac{x}{x^2+1}\right)\mathrm{d}x = \left[\ln(x+1)-\frac{1}{2}\ln(x^2+1)\right]_0^{3/4}$ **[2]**

$= \left[\ln\left(\frac{7}{4}\right)-\frac{1}{2}\ln\left(\frac{9}{16}+1\right)-\ln(1)+\frac{1}{2}\ln(1)\right] = \ln 7-\ln 4-\frac{1}{2}\ln\left(\frac{25}{16}\right)$

$= \ln 7-\ln 4-\ln\left(\frac{5}{4}\right) = \ln 7-\ln 4-\ln 5+\ln 4 = \ln 7-\ln 5 = \ln\left(\frac{7}{5}\right)$. **[M2A2]**

B4(i) $\overrightarrow{AB} = 2\mathbf{i} + 7\mathbf{j} - 8\mathbf{k}$, **[1]**

$\overrightarrow{OC}.\overrightarrow{AB} = 3 \times 2 - 2 \times 7 + 1 \times 8 = 0$ **[1]**

So the vectors are perpendicular. **[1]**

(ii) $L_1 = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - 5\mathbf{j} + 3\mathbf{k})$ **[2]**

(iii) If L_1, L_2 intersect then

$3 + \lambda = 3 - \mu \qquad \lambda + \mu = 0$ **[1]**

$-2 - 5\lambda = 2 + 4\mu \qquad 5\lambda + 4\mu = -4$ **[1]**

$-1 + 3\lambda = -5 - 2\mu \qquad 3\lambda + 2\mu = -4$ **[1]**

Solving these we get $\lambda = -4,\ \mu = 4$ which satisfies all three equations **[2]**

So the lines intersect at $-\mathbf{i} + 18\mathbf{j} - 13\mathbf{k}$. **[1]**

(iv) $(\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}).(-\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = -1 - 20 - 6 = -27$ **[1]**

$|\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}| = \sqrt{1 + 25 + 9} = \sqrt{35}$

$|-\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}| = \sqrt{1 + 16 + 4} = \sqrt{21}$ **[1]**

$\cos\theta = \dfrac{27}{\sqrt{35}\sqrt{21}} \approx 0.9959$

$\theta = 0.0905 = 5.18°$ (Accept either degrees or radians.) **[M1A1]**

B5(a)

(i) $f(x) = e^x - 2x - 1 = -0.0799$ when $x = 1.2$ and $= 0.0693$ when $x = 1.3$. **[2]**

Since the sign changes there must be a root between these points. **[1]**

(ii) $f'(x) = e^x - 2$. **[1]**

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 1.2 - \dfrac{-0.0799}{1.3201} = 1.2605$. **[M1A1]**

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 1.2605 - \dfrac{-0.0062}{1.5272} = 1.256$ **[M1A1]**

(b)(i) $\dfrac{dx}{dt} = -kx$ **[2]**

(ii) $\int \dfrac{dx}{x} = \int -k\,dt$ **[1]**

$\ln x = -kt + c$ **[2]**

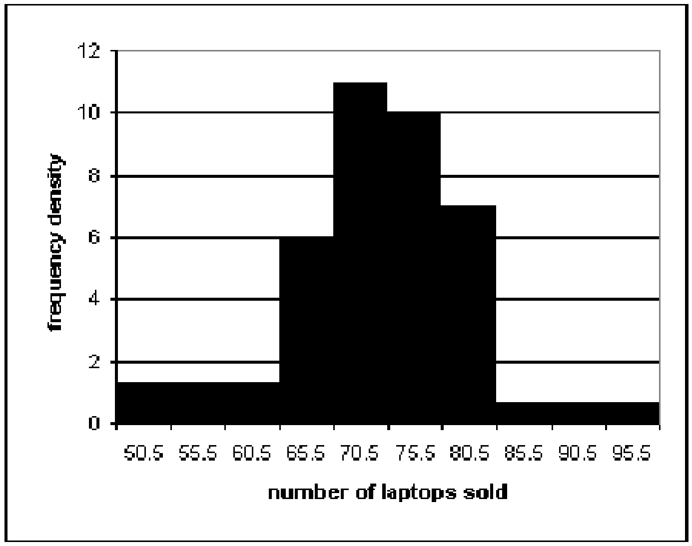
$x = C e^{-kt}$

$a = C e^0 = C$

$x = a e^{-kt}$ **[2]**

B6(i)

from	to	w	f	f/w
45.5	50.5	5	6.67	1.333
50.5	55.5	5	6.67	1.333
55.5	60.5	5	6.67	1.333
60.5	65.5	5	30	6
65.5	70.5	5	55	11
70.5	75.5	5	50	10
75.5	80.5	5	35	7
80.5	85.5	5	3.33	0.667
85.5	90.5	5	3.33	0.667
90.5	95.5	5	3.33	0.667



Labels on horizontal axis. **[1]**
Horizontal scale (accept 60/60.5/61 on boundary). **[1]**
Labels on vertical axis. **[1]**
Vertical scale. **[1]**
Bars right height and width. **[2]**

(ii)

		x	f	xf	xxf
45.5	60.5	53	20	1060	56180
60.5	65.5	63	30	1890	119070
65.5	70.5	68	55	3740	254320
70.5	75.5	73	50	3650	266450
75.5	80.5	78	35	2730	212940
80.5	95.5	88	10	880	77440
			200	13950	986400

x column **[1]**

xf column **[2]**

$$\text{mean} = \frac{\sum xf}{\sum f} = \frac{13950}{200} = 69.75\,.$$ **[1]**

xxf column **[2]**

$$\text{sd} = \sqrt{\frac{\sum x^2 f}{\sum f} - \left(\frac{\sum xf}{\sum f}\right)^2} = \sqrt{\frac{986400}{200} - 69.75^2} = 8.18$$ **[M2A1]**