THE NCUK INTERNATIONAL FOUNDATION YEAR

IFYFM002 Further Mathematics Examination

2017-18

MARK SCHEME

Notice to Markers

This mark scheme should be used in conjunction with the NCUK Centre Marking and Recording results policy, available from the secure area of the NCUK website (http://www.ncuk.ac.uk). Contact your Principal/ Academic Manager if you do not have login details.

Significant Figures:

All correct answers should be rewarded regardless of the number of significant figures used, with the exception of question A1. For this question, 1 discretionary mark is available which will only be awarded to students who correctly give their answer to the number of significant figures explicitly requested.

Error Carried Forward:

Whenever a question asks the student to calculate - or otherwise produce - a piece of information that is to be used later in the question, the marker should consider the possibility of error carried forward (ECF). When a student has made an error in deriving a value or other information, provided that the student correctly applies the method in subsequent parts of the question, the student should be awarded the Method marks for the part question. The student should never be awarded the Accuracy marks, unless a follow through is clearly indicated in the mark scheme. (This is denoted by A1ft or B1ft.) When this happens, write ECF next to the ticks.

M=Method (In the event of a correct answer, M marks can be implied unless the M mark is followed by * in which case, the working must be seen.)
A=Answer
B = Correct answer independent of method
If a student has answered more than the required number of questions, credit should only be given for the first *n* answers, in the order that they are written in the student's answer booklet (*n* being the number of questions required for the examination). Markers should **not** select answers based on the combination that will give the student the highest mark. If a student has crossed out an answer, it should be disregarded.

Section A

Question A1

a) Multiplies out and compares real or imaginary parts **[M1]**

$p = 5$ **[A1]**

b) Finds $\tan^{-1}\left(\frac{1}{12}\right)$ **[M1]**

0.08314… (can be implied) **[A1]**

0.083 to two significant figures. (Allow follow through) **[A1ft]**

Question A2

$3\sum_{r=1}^{n} r = 513$ giving $\frac{3n(n+1)}{2} = 513$ **[M1]**

Forms and solves a quadratic equation **[M1]**

$n = 18$ (ignore any reference to -19) **[A1]**

Question A3

Multiplies through by $(x-5)^2$ **[M1*]**

Reaches a cubic inequality in the form of a product of a linear expression and a quadratic expression $[(x-5)(x^2-4x-12) < 0]$ **[M1*]**

Finds 3 critical values $(-2,\ 5,\ 6)$ **[M1]**

$x < -2$ **(A1)** $x > 5$ $x < 6$ (or $5 < x < 6$) **(A1)** **[A2]**

Please note: the first and second ranges can be separated by a space, a comma or the word 'or'. The final mark is lost if the word 'and' is seen. The second and third ranges, if not given in the form $5 < x < 6$, can be separated by a space or a comma; but in this case the word 'and' is also permitted and the word 'or' will lose the final mark [but not if it has already been lost earlier i.e. a maximum of 1 mark is lost overall].

Special case If the candidate multiplies through by $(x-5)$ and obtains critical values -2 and 6, the only mark available is the second M mark.

Question A4

a) Uses any suitable equation of motion **[M1]**

25 (seconds) **[A1]**

b) Uses any suitable equation of motion **[M1]**

Anything rounding to 3060 (metres) **[A1]**

Question A5

Writes $y\ dx$ as $t^3\ \frac{dx}{dt} \times dt$ **[M1*]**

Writes integral in terms of t with changed limits ($\int_0^2 t^3(2t+1)\,dt$) **[M1*]**

Integrates (sight of a raised index of t is sufficient for this mark). **[M1*]**

$= \frac{84}{5}$ or equivalent. **[A1]**

Question A6

Sets $15 = a\ \times 1\frac{1}{4}$ and finds a value for a $(= 12)$ **[M1]**

$b^2 = \text{their } a^2(\frac{25}{16} - 1)$ and finds a value for b $(= 9)$ **[M1]**

$\frac{x^2}{144} - \frac{y^2}{81} = 1$ or equivalent **[A1]**

Question A7

Finds work done [$1500g\ \times 15 = 220500$ (Joules)] **[M1]**

Divides by 7350 **[M1]**

= 30 (seconds) **[A1]**

Question A8

$f(x) = \tan x$; $f'(x) = \sec^2 x$; $f''(x) = 2\sec^2 x \tan x$; **[M1*]**

$f'''(x) = 2[2\sec^2 x \tan^2 x + \sec^4 x]$

$f(0) = 0$; $f'(0) = 1$; $f''(0) = 0$; $f'''(0) = 2$ **[M1*]**

$\tan x\ \approx x + \frac{x^3}{3}$ (any correct version which does not need to be simplified) **[A1]**

Question A9

$[\boldsymbol{r} - (3\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k})].(2\boldsymbol{i} + 5\boldsymbol{j} - \boldsymbol{k}) = 0$ **[M1]**

$\boldsymbol{r}.(2\boldsymbol{i} + 5\boldsymbol{j} - \boldsymbol{k}) = (3\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}).(2\boldsymbol{i} + \boldsymbol{5j} - \boldsymbol{k})$ **[M1]**

Finds scalar product of RHS (= -8) **[M1]**

$\boldsymbol{r}.(2\boldsymbol{i} + 5\boldsymbol{j} - \boldsymbol{k}) = -8$ **[A1]**

Question A10

Finds integrating factor $e^{\int \sec^2 x \, dx} = e^{\tan x}$ and multiplies through equation **[M1]**

Reaches $\frac{d}{dx}(e^{\tan x} y) = e$ **[M1]**

Integrates both sides ($ye^{\tan x} = ex + c$) with $+ c$ on one side and finds value of c (1 if on RHS) [If the constant is not seen, both this mark and the next are lost.] **[M1]**

$y = \frac{ex + 1}{e^{\tan x}}$ or equivalent. **[A1]**

Question A11

$x = \sinh y$ **[M1*]**

$\frac{dx}{dy} = \cosh y$ **[M1*]**

$= \sqrt{1 + \sinh^2 y} = \sqrt{(1 + x^2)}$ (Writes expression in terms of x). **[M1*]**

$\frac{dy}{dx} = \frac{1}{\sqrt{(x^2 + 1)}}$ (All M marks scored and no errors seen) **[A1]**

Question A12

$m = 110$ **[B1]**

$s = \frac{6.2 \times \sqrt{9}}{1.86}$ **[M1]**

$s = 10$ **[A1]**

Section B

Question B1

a) i. $T - kmg = km \times 1.4$ and $mg - T = m \times 1.4$ **[M1]**

Adds and forms an equation in k. $[(-kmg + mg = 1.4m(k+1)]$ **[M1]**

Solves to find a value for k. **[M1]**

$k = \frac{3}{4}$ or equivalent. **[A1]**

ii. Uses a suitable equation of motion to find the speed of the particles after 3 seconds [4.2 (ms^{-1})] **[M1]**

Uses a suitable equation of motion when string breaks with the initial speed set at their 4.2, final speed = 0 and acceleration = -9.8 (ms^{-2}) **[M1]**

0.9 (metres) **[A1]**

iii. Uses a suitable equation of motion with $t = 4, u =$ their 4.2 and $a = 9.8$ **[M1]**

43.4 (ms^{-1}) **[A1]**

b) i. Finds momentum before collision $[4 \times 5 - 4 \times 1 = 16\ (\mathrm{Ns})]$ **[M1]**

Finds momentum after collision $[5 \times 0.6 + 4 \times v]$, sets expressions equal to each other and solves for v. **[M1]**

$v = 3.25$ (ms^{-1}) or equivalent. **[A1]**

ii. 17 (Ns) **[B1]**

iii. $e = \frac{1.95}{\text{their } v}$ **[M1]**

= 0.6 or equivalent (Allow follow through provided $(0 \leq e \leq 1)$ **[A1ft]**

c) i. Sets potential energy equal to kinetic energy $[6 \times g \times 10 = \frac{1}{2} \times 6 \times v^2]$ **[M1]**

$v = 14$ (ms^{-1}) **[A1]**

ii. Finds work done $(= 588\,J)$ and divides by 20 (29.4) **[M1]**

$6 \times g \times \mu = \text{their } 29.4$ **[M1]**

$\mu = 0.5$ or equivalent **[A1]**

Question B2

a) i.

$$\begin{vmatrix} -1-\lambda & 4 & 0 \\ 1 & -\lambda & -1 \\ 0 & -2 & -1-\lambda \end{vmatrix} = 0$$ **[M1*]**

$(-1-\lambda)[-\lambda(-1-\lambda)-2]$ **(M1*)** $-4[1(-1-\lambda)-0]$ $(+0)$ **(M1*)** **[M2*]**

Factorises to form a product of a quadratic expression and a linear expression in λ $[(-1-\lambda)(\lambda^2+\lambda-6)(=0)]$ **[M1]**

$\lambda = -3, -1, 2.$ **[A1]**

ii. $\lambda = -3$:

$$\begin{aligned} -x + 4y \quad &= -3x \\ x \quad - z &= -3y \\ -2y - z &= -3z \end{aligned}$$ **[M1]**

$(y = z;\ x = -2y)$ Eigenvector is $(-2a\boldsymbol{i} + a\boldsymbol{j} + a\boldsymbol{k})$ **[A1]**

$\lambda = -1$:

$$\begin{aligned} -x + 4y \quad &= -x \\ x \quad - z &= -y \\ -2y - z &= -z \end{aligned}$$ **[M1]**

$(y = 0;\ x = z)$ Eigenvector is $(a\boldsymbol{i} + a\boldsymbol{k})$ **[A1]**

$\lambda = 2$:

$$\begin{aligned} -x + 4y \quad &= 2x \\ x \quad - z &= 2y \\ -2y - z &= 2z \end{aligned}$$ **[M1]**

$y = -\frac{3}{2}z;\ x = -2z)$ Eigenvector is $(-4a\boldsymbol{i} - 3a\boldsymbol{j} + 2a\boldsymbol{k})$ **[A1]**

Part b) is on the next page.

Question B2 – (continued)

b) i. Forms and solves an auxiliary equation $[(m-2)^2 = 0,\ \ m = 2$ (twice) **[M1]**

$y = \ Ae^{2x} + Bxe^{2x}$ **[A1]**

ii. Tries $y = ax + b$ and differentiates ($\frac{dy}{dx} = a, \ \ \frac{d^2y}{dx^2} = 0$) **[M1]**

Substitutes into equation and finds values for a and b $(a = \frac{1}{2},\ b = 2)$ **[M1]**

Particular integral is $\frac{1}{2}x + 2$ **[A1]**

iii. $y = Ae^{2x} + Bxe^{2x} + \frac{1}{2}x + 2;$ $\qquad \frac{dy}{dx} = 2Ae^{2x} + Be^{2x} + 2Bxe^{2x} + \frac{1}{2}$ **[M1]**

Substitutes $x = 0$ into y and finds a value for A (3) **[M1]**

Substitutes $x = 0$ into $\frac{dy}{dx}$ and finds a value for B ($\frac{3}{2}$) **[M1]**

$y = 3e^{2x} + \frac{3}{2}xe^{2x} + \frac{1}{2}x + 2$ **[A1]**

Question B3

a) i. $\frac{2ap}{ap^2} = -\frac{aq^2}{2aq}$ **[M1*]**

$\frac{2}{p} = -\frac{q}{2}$ (Rearranges and simplifies) **[M1]**

$pq = -4$ **[A1]**

ii. $\frac{dy}{dx} = \frac{2a}{2ap}$ $(= \frac{1}{p})$ **[M1*]**

$y - 2ap = \frac{1}{p}(x - ap^2)$ **[M1*]**

Rearranges to form $py = x + ap^2$ **[A1]**

iii. $qy = x + aq^2$ but $q = -\frac{4}{p}$ **[M1]**

$-\frac{4y}{p} = x + \frac{16a}{p^2}$ or in any correct form. **[A1]**

iv. Expresses each equation in terms of y and sets them equal to each other $(\frac{x + ap^2}{p} = \frac{px + \frac{16a}{p}}{-4})$ **[M1]**

Rearranges to find a value for x **[M1]**

$x = -4a$ **[A1]**

b) i. $\frac{dy}{dx} = \frac{-2\sin\theta}{\sec^2\theta}$ **[M1]**

Substitutes $\theta = \frac{\pi}{6}$, inverts and changes sign $(\frac{4}{3})$ **[M1]**

Finds coordinates $[\left(\frac{1}{\sqrt{3}}, \sqrt{3}\right)]$ **[M1]**

$y - \sqrt{3} = \frac{4}{3}(x - \frac{1}{\sqrt{3}})$ or equivalent **[A1]**

ii. Sets $y = 0$ **[M1*]**

Finds a value for x at point X $(\frac{-5}{4\sqrt{3}})$ **[M1*]**

Writes the equation of the parabola in the form $y^2 = 4\left(\frac{5}{4\sqrt{3}}\right)x$ **[M1*]**

Confirms directrix of the parabola is $x = -\frac{5}{4\sqrt{3}}$ **[M1*]**

Goes on to confirm that the directrix passes through point X. **[A1]**

Question B4

a) $8\left(\frac{e^x + e^{-x}}{2}\right) - 4\left(\frac{e^x - e^{-x}}{2}\right) = 7$ **[M1]**

Simplifies to an equation containing e^x and e^{-x} $(2e^x + 6e^{-x} = 7)$ **[M1]**

Multiplies through by e^x and forms a quadratic equation in e^x. **[M1]**

Solves $[\, e^x = \frac{3}{2},\ 2]$ **[M1]**

$x = \ln\left(\frac{3}{2}\right),\ \ \ln 2$ **[A1]**

b) Writes integrand as $\frac{1}{\sqrt{[\,(x+7)^2 + 36]}}$ **[M1*]**

Integrates $[\,\sinh^{-1}\left(\frac{x+7}{6}\right)]$ **[M1*]**

Substitutes limits into their integrated expression and subtracts the right way round. **[M1]**

$\ln(\frac{5 + \sqrt{26}}{3 + \sqrt{10}})$ **[A1]**

c) *Please note: this is a 'show that' question so all working must be seen*.

Writes integrand as $\frac{2}{e^x + e^{-x}}$ **[M1*]**

Multiplies top and bottom by e^x $(\frac{2e^x}{e^{2x} + 1})$ **[M1*]**

Substitutes $u = e^x$ and writes integral in terms of u $(\int \frac{2}{u^2 + 1}\ du)$ **[M1*]**

$= 2\tan^{-1}u\ (+c)$ (this must be seen) $= 2\tan^{-1}(e^x) + c$ **[A1]**

d) Area of base $= OA \times OC$ and applies cross product $(289\boldsymbol{k})$ **[M1*]**

Volume $= \frac{1}{3}(\text{their } 289\boldsymbol{k}).(11\frac{1}{2}\boldsymbol{i} + 3\frac{1}{2}\boldsymbol{j} + 18\boldsymbol{k})$ **[M1*]**

= 1734 (square units) **[A1]**

e) Differentiates $\frac{dr}{dt} = (3t^2 - 3t)\boldsymbol{i} + (t^3 - 4t - 6)\boldsymbol{j} + (8t + 4)\boldsymbol{k}$ **[M1]**

Substitutes $t = 4$ into their expression $(36\boldsymbol{i} + 42\boldsymbol{j} + 36\boldsymbol{k})$ **[M1]**

Finds magnitude $(\sqrt{4356} = 66)$ and multiplies by 0.5 **[M1]**

= 33 (Ns) **[A1]**

Question B5

a) $|z + 3 - 2i| = |z - 1 + i|$ **[M1]**

$(x + 3)^2 + (y - 2)^2 = (x - 1)^2 + (y + 1)^2$ **[M1]**

Expands and simplifies **[M1]**

$8x - 6y + 11 = 0$ or equivalent but must be in this form. **[A1]**

b) i.

$$
\begin{array}{r|l}
 & x^2 - 2x + 5 \\
\hline
x^2 + 9 & x^4 - 2x^3 + 14x^2 - 18x + 45 \\
 & x^4 \qquad\quad + 9x^2 \\
\hline
 & \quad -2x^3 + 5x^2 \\
 & \quad -2x^3 \qquad\quad - 18x \\
\hline
 & \qquad\quad 5x^2 \qquad + 45 \\
 & \qquad\quad 5x^2 \qquad + 45 \\
\hline
 & \qquad\quad \ldots\ldots.. \\
\end{array}
$$

Correct first division **[M1]**

Any correct subsequent division **[M1]**

Correct quotient **[A1]**

ii. $(x^2 + 9)(x^2 - 2x + 5) = 0$ $\quad x = \pm 3i$ **[B1]**

Solves $x^2 - 2x + 5 = 0$ $\quad [\,(x - 1)^2 + 4 = 0, \quad (x - 1)^2 = -4]$ **[M1]**

$x = 1 \pm 2i$ **[A1]**

c) Let $w^5 = 243 = re^{i\theta}$ **[M1]**

Here, $r = 243$ and $\theta = 0$, so $|w| = 243^{1/5} = 3$ **[M1]**

…and arg $w = \frac{0 + 2k\pi}{5}$ **[M1]**

$w = 3$; $w = 3e^{\pm 2\pi/5}$; $w = 3e^{\pm 4\pi/5}$.
(A1) for any two correct; **(A2)** for all correct **[A2]**

d) i. Writes $\cos x + i \sin x = e^{ix}$ and $\cos x - i \sin x = e^{-ix}$ **[M1*]**

adds and subtracts. **[M1*]**

Obtains both results with no errors seen. **[A1]**

ii. $\left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 + \left(\frac{e^{ix} - e^{-ix}}{2}\right)^2$ and expands **[M1*]**
$\left[\frac{e^{2ix} + e^{-2ix} + 2}{4} - \frac{e^{2ix} + e^{-2ix} - 2}{4}\right]$

$= \frac{1}{2} + \frac{1}{2}$ or $\frac{4}{4}$ (this must be seen) $= 1$ **[A1]**

Question B6

a) i. Multiplies out and simplifies **[M1]**

Obtains $\alpha^3 + \beta^3$ **[A1]**

ii. $\alpha + \beta = 4; \quad \alpha\beta = \frac{1}{3}$ **[M1]**

$\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2) = 4[(\alpha + \beta)^2 - 3\alpha\beta] = 60$ **[M1]**

$\alpha^3\beta^3 = (\frac{1}{3})^3 = \frac{1}{27}$ **[M1]**

Writes correct form of equation $[x^2 - 60x + \frac{1}{27} = 0]$ **[M1]**

$27x^2 - 1620x + 1 = 0.$ **[A1]**

b) i. $f(x) = x^{1/3}; \; f'(x) = \frac{1}{3}x^{-2/3}; \; f''(x) = -\frac{2}{9}x^{-5/3}$ **[M1*]**

$f\left(\frac{1}{8}\right) = \frac{1}{2}; \quad f'\left(\frac{1}{8}\right) = \frac{4}{3}; \quad f''\left(\frac{1}{8}\right) = -\frac{64}{9}$ **[M1]**

Writes down any correct form of the expansion **[M1]**

$(x + \frac{1}{8})^{1/3} \approx \frac{1}{2} + \frac{4}{3}x - \frac{32}{9}x^2$ (correct expansion, no need to simplify) **[A1]**

ii. Substitutes $x = \frac{1}{10}$ into their expansion **[M1]**

$\frac{269}{450}$ **[A1]**

c) i. $(y = \frac{1}{\cosh x}) \;\; y = 0$ **[B1]**

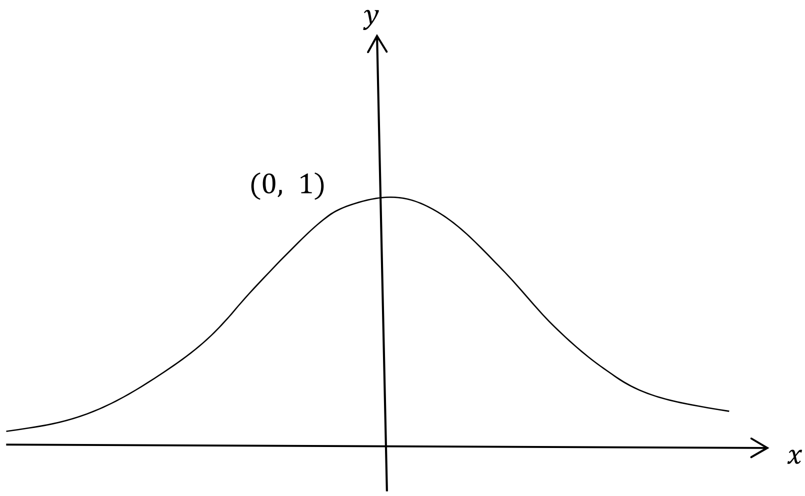
ii. $(0, \; 1)$ **[B1]**

iii. $\frac{dy}{dx}(= \frac{0 - 1 \times \sinh x}{\cosh^2 x}) \quad (= -\operatorname{sech} x \tanh x.)$ **[M1*]**

<u>Either</u> explains that $\cosh x \neq 0$ so $\sinh x = 0$, so $x = 0$
<u>or</u> $\operatorname{sech} x \neq 0$, so $\tanh x = 0$, so $x = 0$ (or similar statement) **[M1*]**

Coordinates are $(0, \; 1)$ **[A1]**

iv.



(B1) for the correct shape which must approach the $x-$ axis asymptotically.

(B1) for $(0, 1)$

Please note: if a candidate uses graph paper, apply no penalty if the graph is clearly a sketch. If there is evidence of plotting points, then take off one mark at the end. **[B2]**

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