THE NCUK INTERNATIONAL FOUNDATION YEAR

IFYFM002 Further Mathematics Examination 2017-18

MARK SCHEME

Notice to Markers

This mark scheme should be used in conjunction with the NCUK Centre Marking and Recording results policy, available from the secure area of the NCUK website (http://www.ncuk.ac.uk). Contact your Principal/ Academic Manager if you do not have login details.

Significant Figures:

All correct answers should be rewarded regardless of the number of significant figures used, with the exception of question A1. For this question, 1 discretionary mark is available which will only be awarded to students who correctly give their answer to the number of significant figures explicitly requested.

Error Carried Forward:

Whenever a question asks the student to calculate - or otherwise produce - a piece of information that is to be used later in the question, the marker should consider the possibility of error carried forward (ECF). When a student has made an error in deriving a value or other information, provided that the student correctly applies the method in subsequent parts of the question, the student should be awarded the Method marks for the part question. The student should never be awarded the Accuracy marks, unless a follow through is clearly indicated in the mark scheme. (This is denoted by A1ft or B1ft.) When this happens, write ECF next to the ticks.

M=Method (In the event of a correct answer, M marks can be implied unless the M mark is followed by * in which case, the working must be seen.)
A=Answer
B = Correct answer independent of method
If a student has answered more than the required number of questions, credit should only be given for the first *n* answers, in the order that they are written in the student's answer booklet (*n* being the number of questions required for the examination). Markers should **not** select answers based on the combination that will give the student the highest mark. If a student has crossed out an answer, it should be disregarded.

Section A

Question A1

a) $p^2 + 8^2 = 17^2$ **[M1]**

$p = \pm 15$ **[A1]**

b) $\tan^{-1}\left(\frac{5}{3}\right)$ **[M1]**

$= 1.0303768$ (Can be implied) **[M1]**

$= 1.0$ to two significant figures. (Allow follow through) **[A1ft]**

Question A2

Det **N** = 1 **[M1*]**

N^{-1} = $\begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix}$ or equivalent **[A1]**

Evidence of any correct method (this must be seen). Sight of the two correct matrices in the product position is good enough. **[M1*]**

M^{T}**N**$^{-1}$ = $\begin{bmatrix} 38 & 31 \\ 7 & 6 \end{bmatrix}$ **[A1]**

Question A3

Applies summation from 1 to 35 and subtracts summation 1 to 19 **[M1*]**

$[3\sum_{r=1}^{35} r^3 - \sum_{r=1}^{35} 11] - [3\sum_{r=1}^{19} r^3 - \sum_{r=1}^{19} 11]$ (Correct breaking up) **[M1*]**

$\left[3 \times \frac{35^2}{4}(35+1)^2 - 11 \times 35\right] - [3 \times \frac{19^2}{4}(19+1)^2 - 11 \times 19]$

At least one intermediate line of working **[M1*]**

Simplifies and reaches a value (1190315 – 108091) **[M1]**

= 1082224 **[A1]**

Question A4

Multiplies both sides by $(x-2)^2$ **[M1*]**

Reaches a cubic inequality in the form of a product of a linear expression and a quadratic expression $[(x-2)(x^2-3x-10) < 0]$ **[M1*]**

Finds 3 critical values $(-2,\ 2,\ 5)$ **[M1]**

$x < -2$ **(A1)** $x > 2 \quad x < 5$ (or $2 < x < 5$) **(A1)** **[A2]**
Please note: the first and second ranges can be separated by a space, a comma or the word 'or'. The final mark is lost if the word 'and' is seen. The second and third ranges, if not given in the form $2 < x < 5$, can be separated by a space or a comma; but in this case the word 'and' is also permitted and the word 'or' will lose the final mark [but not if it has already been lost earlier i.e. a maximum of 1 mark is lost overall].
Special case If the candidate multiplies through by $(x-2)$ and obtains critical values 5 and -2, the only mark available is the second M mark.

Question A5

Uses any suitable equation of motion **[M1]**

Substitutes in and solves **[M1]**

$t = 2.5$ (seconds) (Ignore any reference to 0) **[A1]**

Question A6

Finds the eccentricity $(e = \frac{\sqrt{7}}{4})$ **[M1]**

Gives the coordinates of the foci as $(\pm 12 \times \text{their } e,\ 0)$ and equations of directrices as $x = \pm\frac{12}{\text{their } e}$ **[M1]**

Foci lie at $(\pm 3\sqrt{7},\ 0)$ Directrices have equations $x = \pm\frac{48}{\sqrt{7}}$ or equivalent. **[A1]**

Question A7

Writes $\coth^2 x$ as $\operatorname{cosech}^2 x + 1$ and forms a quadratic equation in $\operatorname{cosech} x$ **[M1*]**
$[\, 8 \operatorname{cosech}^2 x - 6 \operatorname{cosech} x + 1 = 0]$

Solves $[\,(2 \operatorname{cosech} x - 1)(4 \operatorname{cosech} x - 1) = 0\,]$ **[M1]**

Obtains two values of $\operatorname{cosech} x$ ($\frac{1}{2}$ and $\frac{1}{4}$), converts into $\sinh x$ and attempts to express x as a logarithm. **[M1]**

$x = \ln\left(4 + \sqrt{17}\right)$, $x = \ln(2 + \sqrt{5})$ (Both correct) **[A1]**
[Any other valid method is fine]

Question A8

Force acting down slope = $4500g \times \frac{1}{15}$ $(= 2940 \text{ N})$ **[M1]**

Power output = their force × 17 **[M1]**

= anything rounding to 50000 (Watts) **[A1]**

Question A9

Finds $\overrightarrow{OA}$ and $\overrightarrow{OB}$ and then finds $\overrightarrow{OA} \times \overrightarrow{OB}$ **[M1]**

[$\overrightarrow{OA} = 2\boldsymbol{j} - 2\boldsymbol{k}$; $\overrightarrow{OB} = 12\boldsymbol{i} - 8\boldsymbol{j} - 6\boldsymbol{k}$; $\overrightarrow{OA} \times \overrightarrow{OB} = -28\boldsymbol{i} - 24\boldsymbol{j} - 24\boldsymbol{k}$]

Finds magnitude of their $\overrightarrow{OA} \times \overrightarrow{OB}$ and divides by 2. **[M1]**

= 22 **[A1]**

Question A10

$f(x) = (1+x)^{½}$; $f'(x) = \frac{1}{2}(1+x)^{-½}$; $f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$; $f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}}$. **[M1*]**

$f(0) = 1$; $f'(0) = \frac{1}{2}$; $f''(0) = -\frac{1}{4}$; $f'''(0) = \frac{3}{8}$ **[M1]**

$(1+x)^{½} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$ (Accept any correct unsimplified form) **[A1]**

Question A11

Writes integrand as $\frac{1}{\sqrt{[(x+3)^2 - 16]}}$ **[M1*]**

Attempts to integrate $[\cosh^{-1}\left(\frac{x+3}{4}\right)]$ **[M1*]**

Substitutes limits into their integrated expression and subtracts the right way round. **[M1]**

$= \ln\left(\frac{5 + \sqrt{24}}{3 + \sqrt{8}}\right)$ **[A1]**

Question A12

$49 \pm \frac{3 \times 2.201}{\sqrt{12}}$ **[M1]**

47.1 - 50.9 (grams) **[A1]**

Claim reasonable as 50 grams lies in confidence interval. (Allow follow through) **[A1ft]**

Section B

Question B1

a) i. On P: 2g; reaction; frictional force; tension.

On Q: Mg; tension. **(B1)** any 3 correct; **(B2)** all correct. **[B2]**

ii. *Please note: this is a 'show that' question so all working must be seen*.

$Mg - T = Ma$ (One equation correct) **[M1*]**

$T - \frac{1}{2} \times 2g = 2a$ (Second equation correct and then adds) **[M1*]**

$Mg - g = a(M + 2)$ and reaches result with no errors seen. **[A1]**

iii. Substitutes in second equation **[M1]**

$a = 1.4$ (ms^{-1}) **[A1]**

iv. Substitutes in first equation **[M1]**

$M = 1.5$ **[A1]**

b) i. $4u - 5v = -4 \times \frac{3}{4} + 5 \times 1$ and forms an equation in u and v $[4u - 5v = 2]$ **[M1]**

$\frac{7}{20} = \frac{1 + 3/4}{u + v}$ and forms a second equation in u and v $[u + v = 5]$ **[M1]**

Finds one unknown **[M1]**

Finds second unknown **[M1]**

$u = 3;\quad v = 2.$ **[A1]**

ii.

K. E. before collision $= \frac{1}{2} \times 4 \times \text{their } u^2 + \frac{1}{2} \times 5 \times \text{their } v^2$ (28J)

K.E. after collision $= \frac{1}{2} \times 4 \times (\frac{3}{4})^2 + \frac{1}{2} \times 5 \times 1^2$ $(3\frac{5}{8}\text{J})$

One equation correct **(M1)**; second correct and subtracted **(M2)** **[M2]**

Loss $= 24\frac{3}{8}$ (Joules) or answers rounding to 24.4 **[A1]**

Part c) is on the next page.

Question B1 – (continued)

c) Takes moments about some suitable point, Assuming point N:

Clockwise $= 20g \times 4$ Anticlockwise $= 1.5 \times 28g + 0.5 \times xg$ (At least one correct) **[M1]**

Sets equal to each other and solves **[M1]**

$x = 76$ (kg) **[A1]**

Question B2

a) i.

$$\begin{vmatrix} 2-\lambda & -4 \\ -7 & 5-\lambda \end{vmatrix} = 0$$ **[M1*]**

Finds determinant and forms a quadratic equation $[\lambda^2 - 7\lambda - 18 = 0]$ **[M1*]**

$\lambda = -2,\ 9$ **[A1]**

ii.

$\lambda = -2$: **[M1]**
$2x - 4y = -2x$

$-7x + 5y = -2y$ **[A1]**

$(y = x)$ Eigenvector is $(a\boldsymbol{i} + a\boldsymbol{j})$

[M1]

$\lambda = 9$:
$2x - 4y = 9x$

[A1]

$-7x + 5y = 9y$

$(-7x = 4y)$ Eigenvector is $(4a\boldsymbol{i} - 7a\boldsymbol{j})$

b) $(0) - 2(6a - 0) - 3(18 - 0) = -6$ **[M1]**

$a = -4$ **[A1]**

Part c) is on the next page.

Question B2 – (continued)

c) i. Finds an auxiliary equation and attempts to solve it $[m^2 + m - 2 = 0]$ **[M1]**

$y = Ae^{-2x} + Be^{x}$ **[A1]**

ii. Tries $y = a\cos x + b\sin x$ **[M1]**

$\frac{dy}{dx} = -a\sin x + b\cos x$; $\frac{d^2y}{dx^2} = -a\cos x - b\sin x$. **[M1]**

Compares the $\sin x$ and $\cos x$ terms and forms two equations in a and b $[-3a + b = 0;\ -a - 3b = 5]$ **[M1]**

Solves and finds values for a and b $[a = -\frac{1}{2};\ \ b = -\frac{3}{2}]$ **[M1]**

Particular integral is $-\frac{1}{2}\cos x - \frac{3}{2}\sin x$ **[A1]**

iii. $y = Ae^{-2x} + Be^{x} - \frac{1}{2}\cos x - \frac{3}{2}\sin x$; $\frac{dy}{dx} = -2Ae^{-2x} + Be^{x} + \frac{1}{2}\sin x - \frac{3}{2}\cos x$ **[M1]**

Substitutes $x = 0 \text{ and } y = \frac{1}{2}$ into first equation;
substitutes $x = 0 \text{ and } \frac{dy}{dx} = -\frac{13}{2}$ into second equation; and forms two equations in A and B $[A + B = 1;\quad -2A + B = -5]$ **[M1]**

Solves $[A = 2;\ \ B = -1]$ **[M1]**

$y = 2e^{-2x} - e^{x} - \frac{1}{2}\cos x - \frac{3}{2}\sin x.$ **[A1]**

Question B3

a) i. Gradient of chord PQ = $\frac{2aq-2ap}{aq^2-ap^2} = \cdots = \frac{2}{q+p}$ (No need to simplify) **[M1]**

$y-2ap = \frac{2}{q+p}(x-ap^2)$ or $y-2aq = \frac{2}{q+p}(x-aq^2)$ **[A1]**

ii. Substitutes $x = -a$ and $y = 0$ into their equation **[M1*]**

$pq = 1$ **[A1]**

iii. Finds $\frac{dy}{dx}$ $[\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2ap} = \frac{1}{p}]$ **[M1*]**

Inverts and changes sign $[-p]$ **[M1*]**

$y-2ap = -p(x-ap^2)$ or $y = -px+ap^3+2ap$ (or equivalent) **[A1]**

iv. At point R, $y = 0$, so $x = \frac{-ap^3-2ap}{-p} = ap^2+2a$ **[M1]**

Distance = $(ap^2+2a-a) = ap^2+a$ or equivalent **[A1]**

b) i. $\frac{dy}{dx} = \frac{-\csc^2\theta}{-\csc\theta\cot\theta} = \cdots = \sec\theta$ **[M1]**

Substitutes $\theta = \frac{\pi}{3}$ into their $\frac{dy}{dx}$ and their coordinates [2, $\left(\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$] **[M1]**

$y-\frac{1}{\sqrt{3}} = 2(x-\frac{2}{\sqrt{3}})$ or equivalent **[A1]**

ii. Identifies the coordinates of the focus as $(c\sqrt{2}, c\sqrt{2})$ and substitutes these values into their equation of the tangent. **[M1]**

Rearranges to find a value of c. **[M1]**

$c = \frac{3}{\sqrt{6}}$ or equivalent, or anything rounding to 1.22 **[A1]**

iii. Writes integrand as $\frac{1}{\csc^4\theta} \times \frac{dy}{d\theta}d\theta$ **[M1*]**

Changes limits and writes integral in terms of θ [$\int_{\pi/4}^{\pi/6} -\sin^2\theta \ d\theta$] **[M1*]**

Attempts to convert $-\sin^2\theta$ into a double angle [$\frac{1}{2}(\cos 2\theta - 1)$] and integrates [$\frac{1}{2}(\frac{1}{2}\sin 2\theta - \theta)$] **[M1*]**

Substitutes limits into their integrated expression and subtracts the right way round. **[M1]**

$= \frac{\pi}{24}+\frac{\sqrt{3}}{8}-\frac{1}{4}$ or equivalent but must be in exact form. **[A1]**

Question B4

a) $\frac{dy}{dx} = \sinh(\frac{x}{3})$ **[M1*]**

Length of arc = $\int_{\ln 8}^{\ln 27} \sqrt{1 + \sinh^2(\frac{x}{3})}\ dx$ **[M1*]**

Writes $1 + \sinh^2(\frac{x}{3})$ as $\cosh^2(\frac{x}{3})$ and integrates $[3\sinh(\frac{x}{3})]$ **[M1*]**

Substitutes limits into their integrated expression and subtracts the right way round. $[4 - \frac{9}{4}]$ **[M1]**

$= \frac{7}{4}$ or equivalent. **[A1]**

b) *Please note: this is a 'show that' question so all working must be seen*.

$y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

$\frac{dy}{dx} = \frac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2}$ Uses Quotient Rule or other valid method **[M1*]**

At least one intermediate line of working **[M1*]**

Reaches $1 - \frac{\cosh^2 x}{\sinh^2 x}$ **[M1*]**

$= 1 - \coth^2 x$ (this must be seen) = $-\text{cosech}^2 x$. **[A1]**

c) i. The unit vector parallel to plane Π_1 is $\frac{1}{8^2 + (-9)^2 + 12^2}(8\boldsymbol{i} - 9\boldsymbol{j} + 12\boldsymbol{k})$ **[M1]**

The equation of plane Π_1 is $\boldsymbol{r}.\left[\frac{1}{17}(8\boldsymbol{i} - 9\boldsymbol{j} + 12\boldsymbol{k})\right] = \frac{17}{17}$ **[M1]**

Shortest distance is 1 (unit) **[A1]**

ii. Realises that the angle between the planes is the same as the angle between their normals; and finds the scalar product and magnitudes [scalar product is -30; magnitudes are 17 and 7] **[M1]**

Their scalar product = product of their magnitudes $\times \cos\theta$ **[M1]**

Anything rounding to 105 (degrees) or 1.83 (radians). **[A1]**

Part d) is on the next page.

Question B4 – (continued)

d) i. Integrates [$\boldsymbol{v} = (t^4 + c_1)\boldsymbol{i} + (-5t^3 + c_2)\boldsymbol{j} + (3t^2 + c_3)\boldsymbol{k}$] **[M1]**

Substitutes $t = 1$ into their expression and finds values of $c_1,\ c_2$ and c_3 [1, 2 and -2 respectively] **[M1]**

$\boldsymbol{v} = (t^4 + 1)\boldsymbol{i} + (-5t^3 + 2)\boldsymbol{j} + (3t^2 - 2)\boldsymbol{k}$ **[A1]**

ii. Substitutes $t = 2$ into their expression [$17\boldsymbol{i} - 38\boldsymbol{j} + 10\boldsymbol{k}$] and finds magnitude. **[M1]**

$\sqrt{1833}$ or anything rounding to 42.8 (ms^{-1}) **[A1]**

Question B5

a) $\frac{p\, +\, qi}{1\, +\, 2i} = 3 + 5i$ **[M1]**

Multiplies through by $1 + 2i$ and compares real and imaginary parts **[M1]**

$p = -7, \quad q = 11.$ **[A1]**

b) $(x+1)^2 + y^2 = x^2 + (y+1)^2$ and multiplies out **[M1]**

$x = y$ **[A1]**

c) Realises it is a quadratic equation in x^2 **[M1]**

Solves and obtains two values for x^2 $[-\frac{4}{9}$ and $-1]$ **[M1]**

$x = \pm\frac{2}{3}i$ **(A1)** or $\pm i$ **(A1)** **[A2]**

d) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$ **[M1*]**

Expands LHS
$[\cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta\,]$ **[M1*]**

Sets $5\sin\theta$ equal to the imaginary parts and uses $\cos^2\theta = 1 - \sin^2\theta$. **[M1*]**

$\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta.$ **[A1]**

e) i. $|z^3| = 27,\ \arg z^3 = -\pi$ so $z^3 = 27[\cos(-\pi + 2k\pi) + i\sin(-\pi + 2k\pi)]$

(Puts into modulus argument form) **[M1]**

$z = 27^{1/3}[\cos(-\pi + 2k\pi) + i\sin(-\pi + 2k\pi)]^{1/3}$

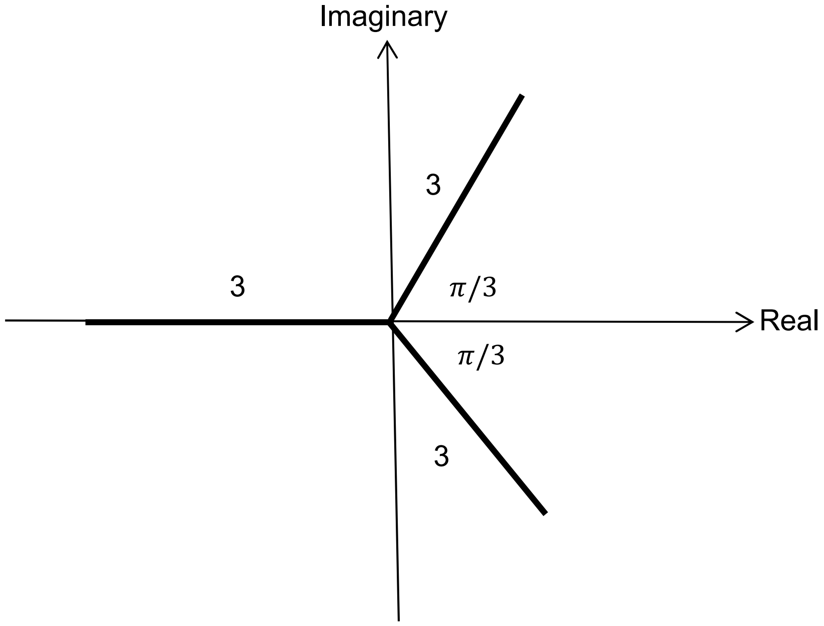
$= 3[\cos\left(\frac{-\pi\, +\, 2k\pi}{3}\right) + i\sin(\frac{-\pi\, + 2k\pi}{3})]$ (Uses De Moivre's Theorem) **[M1]**

Choose a value of k and obtains a solution **[M1]**

Chooses at least one other value of k and obtains a solution **[M1]**

$-3,\ \frac{3}{2} - \frac{3\sqrt{3}}{2}i,\quad \frac{3}{2} + \frac{3\sqrt{3}}{2}i$ **[A1]**

ii. Argand diagram



Lines in correct position **(B1)**

Moduli and arguments shown **(B1)** **[B2]**

Question B6

a) i. $\alpha + \beta = -\frac{k}{4}, \quad \alpha\beta = \frac{5}{4}$ **[M1]**

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{k^2}{16} - \frac{10}{4} = \frac{13}{2}$ **[M1]**

$k = 12$ (Ignore any reference to -12) **[A1]**

ii. $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{-\text{their } k/4}{5/4} \left(= -\frac{12}{5}\right)$; $\quad \frac{1}{\alpha} \times \frac{1}{\beta} = \frac{4}{5}$ **[M1]**

$5x^2 + 12x + 4 = 0$ **[A1]**

b) i. $f(x) = \tan x; \quad f'(x) = \sec^2 x; \quad f''(x) = 2\sec x \times \sec x \tan x$ **[M1*]**

$f\left(\frac{\pi}{4}\right) = 1; \quad f'\left(\frac{\pi}{4}\right) = 2; \quad f''\left(\frac{\pi}{4}\right) = 4$ **[M1*]**

Writes expansion in correct form **[M1]**

$\tan\left(x + \frac{\pi}{4}\right) \approx 1 + 2x + 2x^2$ (Need not be simplified) **[A1]**

ii. Writes 2° in radians $\left(\frac{\pi}{90}\right)$ **[M1]**

Substitutes their $\frac{\pi}{90}$ into their expansion **[M1]**

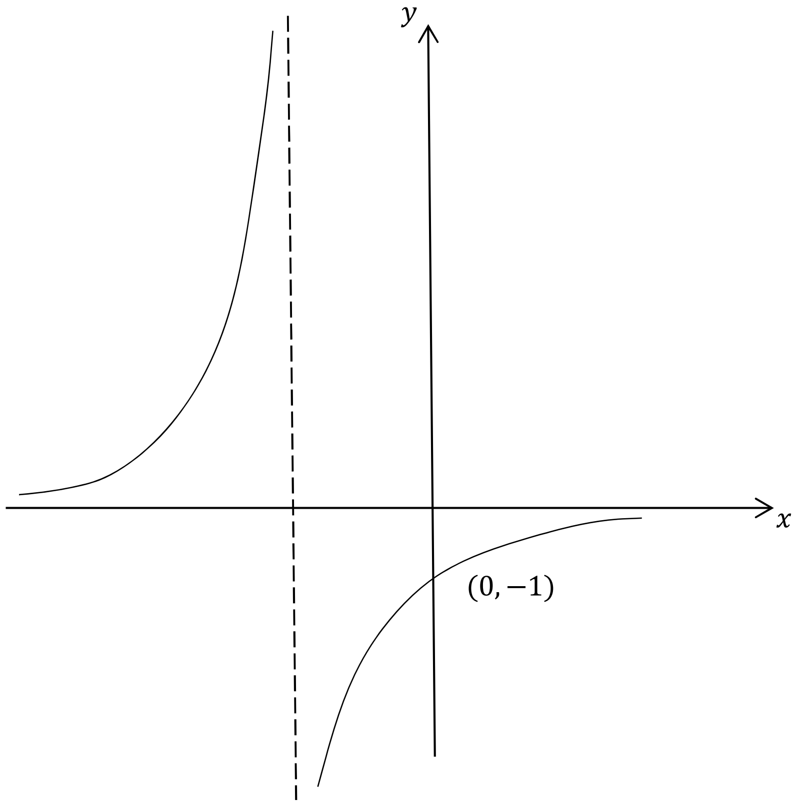
$1 + \frac{\pi}{45} + \frac{\pi^2}{4050}$ **[A1]**

c) i. $x = -1, \quad y = 0$ **(B1)** for each **[B2]**

ii. $(0, -1)$ **[B1]**

iii. Uses Quotient Rule or other valid method $\left[\frac{-(-1)}{(1+x)^2}\right]$ **[M1]**

This is never zero, so there are no stationary values (or similar comment) **[A1]** iv.



(B1) for correct asymptotes* and (0, -1) shown.

[*There is no need to label the $x-$ axis as $y = 0$ as long as the curve clearly approaches it asymptotically.]

(B1) for one correct section of the curve. **(B1)** for second correct section

<u>Please note</u>: if a candidate uses graph paper, apply no penalty if the graph is clearly a sketch. If there is evidence of plotting points, then take off one mark at the end. **[B3]**

Blank Page