THE NCUK INTERNATIONAL FOUNDATION YEAR
IFYFM001 Further Mathematics

2012-13

Mark Scheme

<u>Notice to markers.</u>

Significant Figures:
All <u>correct</u> answers should be rewarded regardless of the number of significant figures used, with the exception of question A6. For this question, 1 discretionary mark is available which will <u>only</u> be awarded to students who correctly give their answer to the number of significant figures explicitly requested.

Error Carried Forward:
Whenever a question asks the candidate to calculate-or otherwise produce-a piece of information that is to be used later in the question, a marker should consider the possibility of error carried forward. A careless error early in the question may make it impossible for a candidate to answer the remainder of the question correctly. Where a candidate has been careless with initial data, but has gone on to demonstrate knowledge of the correct method, they should be awarded marks for the method only.

When this happens, write ECF next to the ticks.

M=Method
A=Answer

Section A

A1

a) $2z - 3w = (2 - 4i) - (9 + 3i) = -7 - 7i$ **1**

b) $i(w - z) = 2i - 3$ and $w^* = 3 - i$ **2**

So we have $\frac{-3+2i}{3-i} = \frac{(-3+2i)(3+i)}{10} = \frac{-11+3i}{10}$ **1**

$= \frac{-11}{10} + \frac{3i}{10}$ **1**

A2

a) Stretch scale factor 3 in the x direction and by 2 in the y direction **1**

b) Stretch scale factor $\frac{1}{3}$ in the x direction and by $\frac{1}{2}$ in the y direction **1**

c) The matrix is $\begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\sqrt{3} \\ \frac{3\sqrt{3}}{2} & 1 \end{pmatrix}$ **3**

(One mark for rotation matrix, one for correct order of multiplication and one for answer)

A3

a) Diagram with weight acting downwards, normal reaction perpendicular to the plane, frictional force acting down the plane. **1**

b) Resolving horizontally X = Fcos30 + R sin30 **1**
Resolving vertically 3g + Fsin30 = Rcos 30 **1**

Since F = μR, F = R/2 **1**

R = $\frac{3g}{cos30-0.5\,sin30}$ so X = $\frac{3g(0.5cos30+sin30)}{cos30-0.5sin30}$ = 44.5N (3sf) **1**

A4

The auxiliary equation is $m^2 + 4 = 0$ and so has roots 2i and – 2i.
Hence the complementary function is y = Acos2x + Bsin2x **1**
The trial function is y = k so y′ = 0 and y′′ = 0.
Substituting gives 4k = 8 so k=2. **1**
So the general solution is y = Acos2x + Bsin2x + 2
Using the condition y=0 when x = $\frac{\pi}{2}$ Gives 0 = -A +0 +2 so A =2. **1**
y′ = -2Asin2x + 2Bcos2x and using y′ =1 when x = $\frac{\pi}{2}$ gives 1 = 0-2B so B = -1/2 **1**

therefore the particular solution is y = 2cos2x -$\frac{1}{2}$ sin2x + 2 **1**

A5

y(0) = 1 **1**
y′(x) = (2x-4) $e^{(x^2-4x)}$ so y′(0) = -4 **1**
y′′(x) = $(2x-4)^2$ $e^{(x^2-4x)}$ + 2$e^{(x^2-4x)}$ so y′′(0) = 16 + 2 = 18 **1**

Therefore the first 3 terms are 1 - 4x + $9x^2$ **1**

A6

a) Using the equation v = $\omega\sqrt{a^2-x^2}$ where a is the amplitude and x the displacement
then 12 = $\omega\sqrt{7}$. So $\omega = 12/\sqrt{7}$ **1**
Maximum velocity is given when x=0 **1**
so v_{max} = $|a|\omega = 48/\sqrt{7}$ (or 18.1 ms^{-1}). **1**

b) Maximum acceleration is given when x =$\pm$a **1**
The magnitude is equal to $|a|\omega^2$ = (4)144/7
= 82.29 ms^{-2} **2**
(1 mark value + 1 mark 4 significant figures only)

A7

a) $\cosh^2 x = (e^{2x} + 2 + e^{-2x})/4$ $\sinh^2 x = (e^{2x} - 2 + e^{-2x})/4$
so $\cosh^2 x - \sinh^2 x$ = ½ - (-1/2) = 1. **1**

b) Using the proven identity gives
5 + 5 $\sinh^2 x$ – 11sinhx -3 = 0 **1**
or (5sinhx – 1)(sinhx - 2) = 0 **1**

So x = $\sinh^{-1}(1/5)$ or x = $\sinh^{-1}(2)$ so

$x = ln\frac{1+\sqrt{26}}{5}$ **1**
or $x = \ln(2+\sqrt{5})$ **1**

A8

a) dy/dt = $-e^{-t}$, dx/dt = $2e^{2t} + 3$ so dy/dx at t= 0 is (-1)/(2+3) = $\frac{-1}{5}$ **1**

b) Area is given by $\int_{t1}^{t2} y\frac{dx}{dt}\,dt$ which is equal to **1**

$\int_0^1 e^{-t}(2e^{2t}+3)dt$ **1**

$= \int_0^1 2e^t + 3e^{-t}\ dt$ **1**

Which is $[2e^t - 3e^{-t}]_0^1$ = $(2e - 3e^{-1}) - (2 - 3) = 2e - 3e^{-1} + 1$ **1**

Section B

B1

a) i) Let the tension be T and the acceleration a.
Applying F= ma for A $T - 2g = 2a$ **1**
Applying F = ma for B $3g - T = 3a$ **1**
Adding gives $a = g/5 = 1.96\text{ms}^{-2}$ **1**

ii) Substituting in the equation for A, $T = 2g/5 + 2g$ **1**
$= 12g/5 = 23.5$ N (3sf) **1**

iii) Velocity of B on hitting the plane is given by $v^2 = 0 + 2as$
so $v = \sqrt{1.96}\ \text{ms}^{-1} = 1.4\ \text{ms}^{-1}$ **1**
When B hits the plane A is 1m above it and moving with this speed upwards
Further distance upwards travelled by A is given by
$0 = (1.4)^2 - 2gs$ so $s = 0.1$m **1**
Hence A's maximum height above the table is $1 + 0.1 = 1.1$ m **1**

b) i) At $t = 1$, $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j}$ so the speed is $\sqrt{20}$ ms^{-1} (or 4.47) **1**

ii) $\boldsymbol{r} = \int \boldsymbol{v}\, dt$ so $\mathbf{r} = (t^3 - t^2/2)\boldsymbol{i} + t^4\boldsymbol{j} + \mathbf{c}$ where $\mathbf{c}$ is a constant vector **1**
Using the initial condition, $3\boldsymbol{i} - 2\boldsymbol{j} = \mathbf{0} + \mathbf{0} + \mathbf{c}$ **1**
Therefore $\mathbf{r} = (t^3 - t^2/2 + 3)\boldsymbol{i} + (t^4 - 2)\boldsymbol{j}$ **1**

iii) P crosses the x axis when the $\mathbf{j}$ component of $\mathbf{r}$ is $\mathbf{0}$ ie when $t = \sqrt[4]{2}$ **1**
The acceleration vector is $\mathbf{a} = (6t - 1)\boldsymbol{i} + 12t^2\boldsymbol{j}$ **1**
The magnitude of the force is $m|\mathbf{a}| = 2\sqrt{6.135^2 + 288} = 36.1$ N (3sf) **1**

B2

a) Area = $\int_1^4 k\sqrt{x}dx$ = $[\frac{2kx^{\frac{3}{2}}}{3}]$ between 1 and 4 = [16k/3 – 2k/3] = $\frac{14k}{3}$ **M1A1**

b) x coordinate of G relative to O is $\frac{\int_1^4 xy\, dx}{A}$ where A is the area **M1**

$= \frac{\int_1^4 kx^{\frac{3}{2}}\, dx}{A}$ = [$\frac{2kx^{\frac{5}{2}}}{5}$]/A between 1 and 4 = [64k/5 – 2k/5]/A = (62k/5)(3/14k)

$= \frac{186k}{70k} = \frac{186}{70}$ **A1**

So the distance from AD is 186/70 – 1 **M1**
$= \frac{116}{70}$

c) y coordinate of G relative to O is ½ $\frac{\int_1^4 y^2\, dx}{A}$ where A is the area **M1**

= ½ $\frac{\int_1^4 k^2x\, dx}{A}$ = ½ [$\frac{k^2x^2}{2}$]/A between 1 and 4 = ($15k^2$/4)/ (14k/3) = $\frac{45k}{56}$ **A1**

d) Let the perpendicular from G to CB meet CB at N
Then CN = $5\sqrt{4}$ – 225/56 = 335/56 **M1**
Also GN = 3 – 116/70 = 94/70 **M1**
If the angle CB makes with the vertical is φ then
tanφ = GN/CN = (94/70) / (335/56) = 0.2244776 **M1**
So φ = 12.7° (3sf) **A1**

e) The critical angle for toppling is given by $\tan^{-1}$ (116/70) / (225/56) **M1**
So tanθ = 0.412 (3sf) **A1**
The angle at which the lamina will start to slide is given by $\tan^{-1}\mu$ **M1**

Therefore if the lamina slides before toppling $\tan^{-1}\mu < \tan^{-1} 0.412$ **M1**
ie if $\mu < 0.412$

B3

a) i) The sum = $\sum_1^n (r^3 + 3r^2 - 3r)$ $= \sum_1^n r^3 + 3\sum_1^n r^2 - 3\sum_1^n r$ **M1**

Which is $\frac{1}{4}n^2(n+1)^2 + \frac{3n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{2}$ **M1A1**

$= \frac{n(n+1)}{4}[n(n+1) + 2(2n+1) - 6]$ **1**

$= \frac{n(n+1)}{4}[n^2 + n + 4n + 2 - 6]$

$= \frac{n(n+1)(n^2+5n-4)}{4}$ **1**

ii) Sum is $\sum_{11}^{30} r(r^2 + 3r - 3)$

$= \sum_1^{30} r(r^2 + 3r - 3)$ - $\sum_1^{10} r(r^2 + 3r - 3)$ **1**

$= \frac{1}{4}$30.31(900 + 150 – 4) - $\frac{1}{4}$ 10.11(100 + 50 - 4) = 239180 **1**

b) Surface area is given by $2\pi\int_1^{15} y\ \sqrt{1 + (\frac{dy}{dx})^2}\, dx$ 1

$\frac{dy}{dx} = \frac{1}{\sqrt{2x+5}}$ **2**

So the area is $2\pi \int_1^{15} \sqrt{2x+5}\ \sqrt{1 + \frac{1}{2x+5}}\ dx$ **1**

Which equals $2\pi \int_1^{15} \sqrt{2x+6}\, dx$ **1**

$= 2\pi\ [\ \frac{1}{3}(2x+6)^{\frac{3}{2}}]$ **1**

evaluated at 1 and 15 **1**

Which gives $\frac{2\pi}{3}[\ 36^{\frac{3}{2}} - 8^{\frac{3}{2}}]$ **1**

$= \frac{2\pi}{3}(216 - 16\sqrt{2})$

B4

a) i) Area = ½ |**AB**x**BC**| = ½ |(**i** -3**j** –**k**)x(**i**+4**j**+**k**)| = ½|**i**-2**j**+7**k**| = ½ $\sqrt{54}$ **M1A1**

ii) A normal to the plane is given by ABxBC = **i** -2**j**+7**k** **1**

so an equation would be r.(**i** -2**j**+7**k**) **= a.n** where **a** is any point on the plane

Using the point *A*, **r**.(**i** -2**j**+7**k**) = (2**i** + **j** + 3**k**).(**i** -2**j**+7**k**) **1**

So **r**.(**i**-2**j**+7**k**) = 21 (or valid alternative) **1**

iii) Distance is $\frac{d}{|\boldsymbol{n}|} = \frac{21}{\sqrt{54}}$ **M1A1**

iv) Volume = 1/3(base)(perpendicular height) = (1/3)(½ $\sqrt{54}$)($\frac{21}{\sqrt{54}}$) = $\frac{7}{2}$

(or use of scalar triple product for example $\frac{1}{6}$ **OA**.**OB**x**OC**

= $\frac{1}{6}$(2**i**+**j**+3**k**)**.**(**-**10**i** - **j**+14**k**) **=** 7/2) **M1A1**

b) i) $\alpha + \beta + \gamma = 3, \alpha\beta + \beta\gamma + \gamma\alpha = -5, \alpha\beta\gamma = -3$ **2**

(One or two correct 1 mark, all correct 2 marks)

ii) $\alpha^2 + \beta^2 + \gamma^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$ **M1**

$(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha) = \sum \alpha^2\beta + 2\alpha\beta\gamma$ **M1**

$= (\sum \alpha)(\sum \alpha\beta) - 3\alpha\beta\gamma + 2\alpha\beta\gamma$ **M1**

So the value is 9 + 10 -15 +3 = 7 **A1**

B5

a) i) Another root is 2 -3i **1**

ii) $z^2 - 4z + 13$ must be a factor of the quartic **1**

So $(z^2 - 4z + 13)(az^2 + bz + c) \equiv 4z^4 - 24z^3 + 89z^2 - 124z + 65$ **M1**

Comparing coefficents gives a = 4, c = 5, b – 4a = -24 so b = -8 **1**

So the other two roots are solution of $4z^2 - 8z + 5 = 0$

Hence $z = 1 \pm \frac{1}{2}i$ **A1**

b) i) Accurate diagram (crossing the axes where it should) **1**

Circle centre (-2, 1) **1**

radius 3 **1**

ii) Accurate diagram(only half line and line below the real axis) **1**

Half line starting at (-3,0) **1**

inclined at $\frac{\pi}{6}$ below the real axis **1**

c) We have arg $\frac{x+i(y-3)}{x+i(y+3)} = \frac{\pi}{4}$

Multiplying by the conjugate of the denominator gives

$$\arg\frac{[x+i(y-3)][x-i(y+3)]}{x^2+(y+3)^2} = \frac{\pi}{4}$$ **1**

$$\arg\frac{x^2+(y-3)(y+3)+i\,(yx-3x-yx-3x)}{x^2+(y+3)^2} = \frac{\pi}{4}$$ **1**

Since the argument is $\frac{\pi}{4}$ the real and imaginary parts must be equal **M1**

So $x^2 + y^2 - 9 = -6x$ therefore $x^2 + 6x + y^2 = 9$ **A1**

B6

a) F is at (3, 0) **1**

b) Directrix is at $x = -3$ **1**

c) $dy/dx = (dy/dt)/(dx/dt) = 6/6t = 1/t$ (or equivalent) **1**

So the equation of the tangent is $(y - 6t) = (1/t)(x - 3t^2)$ **1**

Which gives $yt = x + 3t^2$ **1**

d) Coordinates of M are $(-3, 6t)$ **1**

Cordinates of T $(-3t^2, 0)$ **1**

Gradient of FP $= (6t - 0)/(3t^2 - 3) = \frac{2t}{t^2-1}$ **1**

Gradient of TM $= (6t - 0)/(-3-(-3t^2)) = \frac{2t}{t^2-1}$ **1**

Hence they are parallel.

e) $FP^2 = (3- 3t^2)^2 + (0- 6t)^2 = 9(1 - t^2)^2 + 36t^2 = 9(1 - 2t^2 + t^4 + 4t^2) = 9(t^2+1)^2$ **2**

$PM^2 = (3t^2 - (-3))^2 + (6t - 6t)^2 = 9(t^2 + 1)^2$ **1**

Hence they are equal.

f) Since PM is parallel to FT and TM is parallel to FP, and FP = PM, TFPM is a rhombus. **1**

Therefore area of PFM = ½ (area of rhombus) = ½ (base) (perpendicular height) **1**

$= ½ (3t^2 + 3)(6t) = 90$ **1**

(or any other valid method)