THE NCUK INTERNATIONAL FOUNDATION YEAR
IFYFM001 Further Maths

2011-12

Mark Scheme

Section A

Advice to Markers about expressing numerical answers in the correct units and, where appropriate, to an accuracy of three significant figures.

Marks should not be deducted ***from individual answers*** for failure to express numerical answers to 3 significant figures, where this is appropriate, or for failure to use correct units.

Instead the paper should be marked initially without regard to these issues. Then the markers should make a judgement about units and significant figures for the paper ***as a whole*** and then deduct up to 4 marks from the overall total mark.

A1

a) $\frac{1}{z-w} = \frac{1}{-1+3i}$ **1**

$= \frac{-1-3i}{1+9} = \frac{-1}{10} - \frac{3i}{10}$ **1**

b) 3p + 4q = 0 and 2p + q = 3 **M1**

So $p = \frac{12}{5}$ and $q = \frac{-9}{5}$ **A1A1**

A2

a) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **1**

b) Stretch scale factor 3 in x direction (or parallel to x axis) **2**

c) $AB^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \frac{1}{3} & 0 \end{pmatrix}$ (or $B^{-1}A$) **1**

So $AB^{-1}A = \begin{pmatrix} 0 & -1 \\ \frac{1}{3} & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \frac{-1}{3} \end{pmatrix}$ **1**

A3

Total mass is 4x3ρ + 0.5x4x2ρ = 16ρ **1**

CoM rectangle (4, 1.5) CoM triangle $\left(\frac{13}{3}, \frac{11}{3}\right)$ **1**

$16\rho\bar{x} = 12 \text{ x } 4\rho + 4 \text{ x } \frac{13}{3}\rho \quad 16\rho\bar{y} = 12 \text{ x } 1.5\rho + 4\text{x}\frac{11}{3}\rho$ **M1**

So $\bar{x} = \frac{49}{12}$ and $\bar{y} = \frac{49}{24}$ **A1A1**

A4

Integrating factor is $\exp(\int \frac{2}{x}\,dx) = \exp(\ln(x^2)) = x^2$ **1**

So the DE is now $x^2\frac{dy}{dx} + 2xy = \frac{1}{x}$ **1**

Therefore $(yx^2)' = \frac{1}{x}$

So, integrating gives $(yx^2) = \ln(x) + C$ **1**

From the condition given 2 = 0 + C so C=2 **1**

Solution is $y = \frac{\ln x}{x^2} + \frac{2}{x^2}$ **1**

A5

a) The direction of the line is given by **<**3, -2, 2 **>**x**<**1, 2, 3**>** **M1**

= **i**(-6-4) - **j**(9-2) +**k**(6- -2) = -10**i** – 7**j** + 8**k**

Any point on the line is given by the simultaneous solution of

3x - 2y + 2z = 4 and x + 2y + 3z = 11 **1**

Letting x = 0 and adding gives 5z = 15, so z = 3, y = 1

So an equation of the line is

r = **j** + 3**k** + μ(-10**i** – 7**j** + 8**k)** (or equivalent) **1**

b) Angle is given by the angle between the normals

$\cos\theta = \frac{3-4+6}{\sqrt{17}\sqrt{14}}$ so θ = 71.1° **M1A1**

A6

a) F = ma towards centre $\frac{mv^2}{r}$ = Lsinθ **M1**

Resolving vertically mg = Lcosθ **M1**

Dividing gives $\frac{v^2}{rg}$ = tanθ so $\theta = \arctan\frac{200^2}{9.8x3300}$ $\theta = 51.0°$ **A1**

b) L = $\frac{mg}{cos\theta}$ **1**

$= \frac{9.8x2.5x10^5}{0.628717}$ = 3897000 N (4sf only) **1**

A7

The sum is equal to $\sum_{20}^{50} 6n^2 + 7n - 3$ **1**

= n(n+1)(2n+1) +$\frac{7}{2}$n(n+1) -3n **M1A1**

Evaluated between 19 and 50 **M1**

= [50.51.101 + $\frac{7}{2}$50.51 – 3.50] – [19.20.39 + $\frac{7}{2}$19.20 -3.19] = 250232 **1**

A8

$\alpha + \beta = \frac{-b}{a}$ $\alpha\beta = \frac{c}{a}$ **2**

$(1-3\alpha^3)(1-3\beta^3)$ = 1 – 3(α^3+β^3) +9 $\alpha^3\beta^3$

= 1 – 3[$(\alpha+\beta)^3$ - 3αβ(α+β)] + 9$(\alpha\beta)^3$ Symmetric form **1**

= 1 – 3[$\frac{-b^3}{a^3}$ - $\frac{-3bc}{a^2}$] + $\frac{9c^3}{a^3}$ correct substitution **1**

= 1 + $\frac{3b^3}{a^3} - \frac{9abc}{a^3} + \frac{9c^3}{a^3} = \frac{a^3+3b^3+9c^3-9abc}{a^3}$ **1**

Section B

B1

a) i) Using $v^2 = u^2 + 2as$, $v = 0$, $a = -g$, $u = 10$ **1**

gives $s = \frac{100}{2g} = 5.10$m **1**

ii) Applying F = ma upwards

$-mg - \frac{v2}{400} = ma = mv\frac{dv}{dx}$

M1A1

So $0.1g - \frac{v2}{400} = 0.1v\frac{dv}{dx}$

Therefore $-40g - v^2 = 40\frac{dv}{dx}$ **1**

iii) Separating the variables $\frac{40v}{40g+v^2}dv = -dx$ **1**

Integrating gives $\frac{1}{20}\ln(40g + v^2) = -x + C$

1

Since x = 0 when v = 10, $C = \frac{1}{20}\ln(40g + 100)$ **1**

Rearranging, $x = -20ln\frac{40g+v^2}{40g+100}$ **1**

When v = 0, $x = -20ln\frac{40g}{40g+100}$ = 4.54m **1**

b) i) Let v_A and v_B be the velocities of A and B after the impact and taking the direction of A's motion to be positive. Also, assuming v_A and v_B are positive after the impact,

CLM gives $3 \times 4 - 6 \times 3 = 3v_A + 6v_B$ or $-2 = v_A + 2v_B$ **M1**

NLR gives $e = \frac{v_B - v_A}{7}$ or $7e = v_B - v_A$ **M1A1**

Solving simultaneously gives $v_B = \frac{7e-2}{3}$ which is positive if $e > \frac{2}{7}$ **1**

ii) $v_B = ½$ so $v_A = -3ms^{-1}$ or $3ms^{-1}$ (either) **1**

B2

a) Diagram with,

Normal reaction force at B acting away from vertical wall **1**

Normal reaction force at A acting away from horizontal floor **1**

Frictional force at A acting towards vertical wall **1**

Weight force acting downwards *through G* **1**

b) Let the reaction force at A be R, reaction force at B be S, frictional force is F. Coefficient of friction is μ.

Resolving vertically R = mg = 20g **M1**

Resolving horizontally F = S **M1**

Since F = μR, S = μ20g **M1**

Moments about A gives AG(cos30)20g = 8(sin30)μ20g **M1**

So AG = 8(tan30)μ $= \frac{8}{\sqrt{3}}\,\frac{\sqrt{3}}{4} = 2\text{m}$ 1

c) With the same notation
Resolving vertically R = 20g + 60g = 80g **M1**
Resolving horizontally F = S **1**
When the ladder is about to slip F = μR so S = μ80g **1**

Let x be the distance that the man reaches from A

Moments about A gives

AG(cos60)20g + x(cos60)60g = 8(sin60)S **M1A1**

So 40g(cos60) + 60gx(cos60) = 8(sin60)μ80g

2 + 3x = 8(tan60)4μ

x = $\frac{22}{3}$m = 7.33m **A1**

B3 a) i) $\frac{dy}{dx} = 3sech^2 3x$ or $\frac{3}{cosh^2 3x}$ **2**

ii) cosh(ln2) = $\frac{e^{ln2}+e^{-ln2}}{2} = \frac{2+\frac{1}{2}}{2} = \frac{5}{4}$ **1**

iii) When x = $\frac{1}{3}ln2,\ y = \tanh(ln2) = \frac{e^{2ln2}-1}{e^{2ln2}+1} = \frac{3}{5}$ **M1A1**

$\frac{dy}{dx}$ at x = $\frac{1}{3}ln2$ is equal to $\frac{3}{cosh^2 3x} = \frac{3}{\left(\frac{25}{16}\right)} = \frac{48}{25}$ **1**

So the equation of the tangent is

$(y - \frac{3}{5}) = \frac{48}{25}(x - \frac{1}{3}ln2)$ **M1**

Or 25y = 48x + 15 - 16ln2 **A1**

b) $\int_{-1}^{0} \frac{dx}{\sqrt{3x^2+12x+11}} = \frac{1}{\sqrt{3}}\int_{-1}^{0} \frac{dx}{\sqrt{x^2+4x+\frac{11}{3}}}$ **1**

Completing the square of the quadratic gives

$\frac{1}{\sqrt{3}}\int_{-1}^{0} \frac{dx}{\sqrt{(x+2)^2-\frac{1}{3}}}$ **M1A1**

$= \frac{1}{\sqrt{3}}\left[cosh^{-1}(x+2)\sqrt{3}\,\right]\,\Big|_{-1}^{0}$ **M1**

This is equal to $\frac{1}{\sqrt{3}}$ ($\cosh^{-1} 2\sqrt{3}$ – $\cosh^{-1}\sqrt{3}$)

In logarithm form $\frac{1}{\sqrt{3}}\big(\ln(2\sqrt{3} + \sqrt{12-1}\,)$ - $\ln(\sqrt{3} + \sqrt{2})\,\big)$ **M1A1**

Which is equal to

$\frac{1}{\sqrt{3}}\ ln\left(\frac{2\sqrt{3}+\sqrt{11}}{\sqrt{3}+\sqrt{2}}\right)$ or $\frac{1}{\sqrt{3}}\ ln\left(\sqrt{33}-\sqrt{22}-2\sqrt{6}+6\right)$ **A1**

B4 a) The auxiliary equation is $m^2 + 6m + 9 = 0$

This has a double root $m = -3$ **1**

So the complementary function is $y = (Ax + B)e^{-3x}$ **M1**

For the particular integral we use the trial function $y = px + q$ **M1**

Hence $y' = p$ and $y'' = 0$

Substituting gives $0 + 6p + 9(px + q) \equiv 18x + 21$

Therefore $p = 2$ and $q = 1$ **1**

General solution is $y = (Ax + B)e^{-3x} + 2x + 1$

$x = 0$, $y = 0$ implies $0 = B + 1$ so $B = -1$ **1**

$\frac{dy}{dx} = (Ax + B)(-3e^{-3x}) + e^{-3x}A + 2$

$\frac{dy}{dx} = 6$, $x = 0$ implies $6 = -3B + A + 2$, therefore $A = 1$ **1**

Particular solution is $y = (x - 1)e^{-3x} + 2x + 1$ **A1**

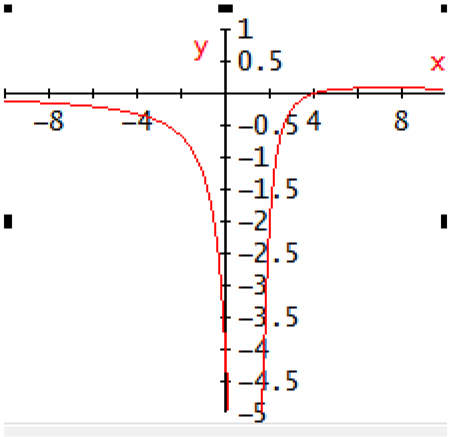
b) i) Horizontal asymptote at $y = 0$ **1**

<u>Double</u> vertical asymptote at $x = 1$ **2**

ii) $x = 0$ implies $y = -4$ so $(0, -4)$ **1**

$y = 0$ implies $x = 4$ so $(4, 0)$ **1**

iii)



Shape and position $x < 1$ **1**
Shape and position $1 < x < 4$ **1**
Shape and position $x > 4$ **1**

Must mention 'double asymptote' somewhere for 3 marks part i) otherwise 2 marks.

B5 a) i) $z^{-1} = \dfrac{1}{\cos\theta + i\sin\theta} = \dfrac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta - i\sin\theta$ **2**

ii) $(2\cos\theta)^5 = (z + \frac{1}{z})^5$ and expanding both sides gives

$$32\cos^5\theta = z^5 + 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) + 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) + \left(\frac{1}{z^5}\right)$$ **M1**

$$= z^5 + \left(\frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$$

$32\cos^5\theta = 2\cos5\theta + 10\cos3\theta + 20\cos\theta$

So $16\cos^5\theta = \cos5\theta + 5\cos3\theta + 10\cos\theta$ (*) **A1**

Similarly

$(2\cos\theta)^3 = (z + \frac{1}{z})^3$

$$8\cos^3\theta = z^3 + 3z^2\left(\frac{1}{z}\right) + 3z\left(\frac{1}{z^2}\right) + \left(\frac{1}{z^3}\right)$$ **M1**

$$= z^3 + \left(\frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)$$

So $8\cos^3\theta = 2\cos3\theta + 6\cos\theta$

$16\cos^3\theta = 4\cos3\theta + 12\cos\theta$ (**) **A1**

(*) + (**) gives

$16(\cos^5\theta + \cos^3\theta) = \cos5\theta + 9\cos3\theta + 22\cos\theta$ **M1**

b) i) Modulus is $\sqrt{64 + 192} = 16$ **1**

Argument is $\frac{-\pi}{3}$ **1**

ii) $z^3 = 16[\cos\left(\frac{-\pi}{3} + 2n\pi\right) + i\sin\left(\frac{-\pi}{3} + 2n\pi\right)]$ **M1**

Taking cube roots and using deMoivre's theorem

$z = \sqrt[3]{16}\,[\cos\left(\frac{-\pi}{9} + \frac{2n\pi}{3}\right) + i\sin\left(\frac{-\pi}{9} + \frac{2n\pi}{3}\right)]$ $n = 0, 1, 2$ **M1A1**

n = 0 gives $\sqrt[3]{16}\,[\cos(\frac{-\pi}{9}) + i\sin\left(\frac{-\pi}{9}\right)]$ z = 2.37 - 0.86i **A1**

n = 1 gives $\sqrt[3]{16}\,[\cos(\frac{5\pi}{9}) + i\sin\left(\frac{5\pi}{9}\right)]$ z = -0.44 +2.48i **A1**

n = 2 gives $\sqrt[3]{16}\,[\cos(\frac{11\pi}{9}) + i\sin\left(\frac{11\pi}{9}\right)]$ z = -1.93 -1.62i **A1**

B6 a) $\frac{dx}{dt} = 5$, $\frac{dy}{dt} = \frac{-5}{t^2}$ so $\frac{dy}{dx} = \frac{-1}{t^2}$ **M1**

So the equation of the tangent is given by

$(y - \frac{5}{t}) = \frac{-1}{t^2}(x - 5t)$ **M1A1**

Which gives
$t^2y - 5t = -x + 5t$ or
$t^2y + x = 10t$ **A1**

b) Gradient of the normal is given by t^2 **1**
So the equation of the normal is

$(y - \frac{5}{t}) = t^2(x - 5t)$ **1**

Which gives
$ty - 5 = t^3x - 5t^4$ or
$ty - t^3x = 5(1 - t^4)$ **1**

c) Tangent intersects at (a, 0) implies

$0 + a = 10t$ i.e. $t = \frac{a}{10}$ **M1**

So P has coordinates $\left(\frac{5a}{10}, \frac{5x10}{a}\right)$ therefore P is $\left(\frac{a}{2}, \frac{50}{a}\right)$ **A1**

d) Q has x coordinate a and lies on the hyperbola so its y coordinate is $\frac{25}{a}$
So Q is the point $\left(a, \frac{25}{a}\right)$ **1**

e) Since OQ passes through the origin its gradient is $\frac{25}{a^2}$ **M1**

Gradient of AP is the gradient of the tangent which is $\frac{-1}{t^2} = \frac{-100}{a^2}$ **M1**

OQ and AP perpendicular implies $\frac{25}{a^2} . \frac{-100}{a^2} = -1$

So $a^4 = 2500$ therefore $a = 5\sqrt{2}$ **A1**

f) $PQ^2 = \left(a - \frac{a}{2}\right)^2 + \left(\frac{25}{a} - \frac{50}{a}\right)^2 = \frac{a^2}{4} + \frac{625}{a^2} = \frac{50}{4} + \frac{625}{50} = 25$ **M1**

Hence PQ = 5. **A1**