THE NCUK INTERNATIONAL FOUNDATION YEAR (IFY)

Further Mathematics

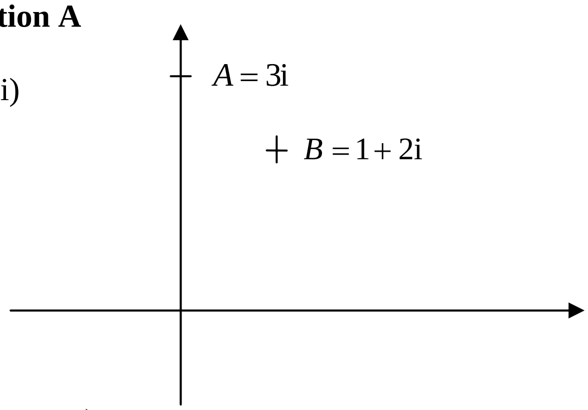
Mark Scheme

Level of accuracy: If a question specifies how many decimal places or significant figures are required, there is a mark for this. Otherwise accept any reasonable level of accuracy and alternative form.

Error carried forward: Where numerical errors have been made, students lose a mark/marks at that stage but may be awarded marks for using correct methods subsequently if the student demonstrates basic understanding.

Section A

A1(i) **2**



(ii) $\overrightarrow{AB} = 1 + 2i - 3i = 1 - i$ **1**

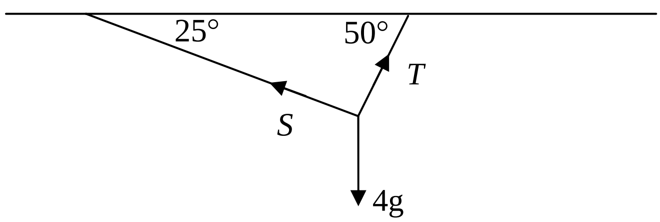
$|AB| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ **1**

$\angle AOB = \arg A - \arg B = 90° - \tan^{-1} 2 = 90° - 63.4° = 26.6° \; (= 0.464\,\text{rad})$ **1**

A2(i) $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$ **2**

(ii) $\mathbf{C} = \mathbf{BA} = \begin{pmatrix} 0.28 & 0.96 \\ -0.96 & 0.28 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0.84 & -0.96 \\ -2.88 & -0.28 \end{pmatrix}$ **M1A2**

A3



Let the tensions in the strings be S and T respectively.

Resolving horizontally: $S\cos 25° = T\cos 50°$. **1**

Resolving vertically: $S\sin 25° + T\sin 50° = 4g$ **1**

So $S\sin 25° + \dfrac{S\cos 25° \sin 50°}{\cos 50°} = 4g$

$S(\sin 25° \cos 50° + \cos 25° \sin 50°) = 4g\cos 50°$

$S = \dfrac{4g\cos 50°}{\sin 75°} = 26.1\text{N}, \qquad T = \dfrac{4g\cos 25°}{\sin 75°} = 36.8\text{N}$ **M1A2**

A4 The integrating factor is $e^{\int -2\mathrm{d}x} = e^{-2x}$ **1**
Multiplying through by this we get
$$e^{-2x}\frac{\mathrm{d}y}{\mathrm{d}x} - 2e^{-2x}y = e^{x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-2x}y\right) = e^{x}$$ **1**
Integrating this: $e^{-2x}y = e^{x} + c$ where c is a constant **1**
So $y = e^{3x} + ce^{2x}$ **1**
From the initial condition $4 = 1 + c$, $c = 3$
Final solution is $y = e^{3x} + 3e^{2x}$ **1**

A5(i) A vector perpendicular to Π is $3\mathbf{i} - 5\mathbf{j} + \mathbf{k}$
so the parametric equation of l is $\mathbf{r} = 7\mathbf{i} - 7\mathbf{j} + t(3\mathbf{i} - 5\mathbf{j} + \mathbf{k})$ **1**
(ii) Substituting in the equation of Π,
$$3(7 + 3t) - 5(-7 - 5t) + (t) + 14 = 0$$
$$(9 + 25 + 1)t + 21 + 35 + 14 = 0$$
$$35t + 70 = 0$$
$$t = -2$$ **1**
The point B has position vector $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ **1**
(iii) The vector $\overrightarrow{AB}$ is $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} - (7\mathbf{i} - 7\mathbf{j}) = -6\mathbf{i} + 10\mathbf{j} - 2\mathbf{k}$ **1**
So the distance AB is $\sqrt{36 + 100 + 4} = \sqrt{140} \approx 11.8$ **1**

A6 Using the usual notation for simple harmonic motion:
$v^2 = \omega^2\left(a^2 - x^2\right)$ and $\ddot{x} = -\omega^2 x$,
the maximum speed occurs when $x = 0$, so $7^2 = \omega^2 a^2$. **1**
and the maximum acceleration occurs when $x = -a$,
and so $15 = \omega^2 a$. **1**
Then the amplitude $a = \dfrac{\omega^2 a^2}{\omega^2 a} = \dfrac{7^2}{15} = \dfrac{49}{15} \approx 3.27\,\text{m}$. **1**
The frequency $\omega = \dfrac{\omega^2 a}{\omega a} = \dfrac{15}{7}$. **1**
The period is $\dfrac{2\pi}{\omega} = \dfrac{14\pi}{15} \approx 2.93\,\text{s}$. **1**

A7 $y = e^{2x}\cos x$ so when $x = 0$, $y = 1$ **1**
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2e^{2x}\cos x - e^{2x}\sin x$ so when $x = 0$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2$ **1**
$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4e^{2x}\cos x - 2e^{2x}\sin x - 2e^{2x}\sin x - e^{2x}\cos x$ so when $x = 0$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 3$ **2**
The Maclaurin series is therefore $1 + 2x + \dfrac{3}{2}x^2 + \ldots$ **1**

A8 $x = 3\cosh\phi$ so $\dfrac{\mathrm{d}x}{\mathrm{d}\phi} = 3\sinh\phi$ and $y = 5\sinh\phi$ so $\dfrac{\mathrm{d}y}{d\phi} = 5\cosh\phi$ **1**

At the point P $x = 3\cosh(\ln 2) = 3\dfrac{\mathrm{e}^{\ln 2} + \mathrm{e}^{-\ln 2}}{2} = 3\dfrac{2+\frac{1}{2}}{2} = \dfrac{15}{4}$ **1**

$y = 5\sinh(\ln 2) = 5\dfrac{\mathrm{e}^{\ln 2} - \mathrm{e}^{-\ln 2}}{2} = 5\dfrac{2-\frac{1}{2}}{2} = \dfrac{15}{4}$ **1**

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}\phi} \Big/ \dfrac{\mathrm{d}x}{\mathrm{d}\phi} = \dfrac{5\cosh\phi}{3\sinh\phi} = \dfrac{25}{9}$ **1**

So the equation of the tangent is $y - \dfrac{15}{4} = \dfrac{25}{9}\left(x - \dfrac{15}{4}\right)$ **1**

Alternative equation: $25x - 9y = 60$

Section B

B1(i)	$a = \frac{\mathrm{d}v}{\mathrm{d}t}$	**1**
	$= \frac{\mathrm{d}v}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}t}$	**1**
	$= v\frac{\mathrm{d}v}{\mathrm{d}x}$	**1**
	$= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right)$	**1**
(ii)	$F = ma$, so $3a = 4 - \sin 5x$	**1**
	$3\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = 4 - \sin 5x$	
	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = \frac{1}{3}(4 - \sin 5x)$	**1**
	$\frac{1}{2}v^2 = \frac{1}{3}\left(4x + \frac{1}{5}\cos 5x\right) + c$	**1**
	When $x = 0$, $v = 3$, so $\frac{9}{2} = \frac{1}{15} + c$ $\quad c = \frac{9}{2} - \frac{1}{15} = \frac{133}{30}$	**1**
	$v^2 = \frac{2}{15}(20x + \cos 5x) + \frac{133}{15}$	**1**
	When $x = 3$, $v = \sqrt{\frac{2}{15}(60 + \cos 15) + \frac{133}{15}} = 4.0946 \approx 4.09\,\mathrm{ms}^{-1}$	**2**
(iii)	Before the collision the momentum is $3v$.	**1**
	After the collision the momentum is $(3+7)v' = 10v'$	**1**
	By conservation of momentum $3v = 10v'$	**1**
	So after the collision $v' = \frac{3}{10}v \approx 1.23\,\mathrm{ms}^{-1}$.	**1**

B2(i) Taking the x-axis as lying up the plane and the y-axis perpendicular to the plane, and letting the density of the lamina be ρ kgm^{-2}, the mass of the larger rectangle is 0.24ρ and that of the smaller rectangle is 0.08ρ. **2**

Thus the centre of gravity of the lamina is $G = (\bar{x}, \bar{y})$, where

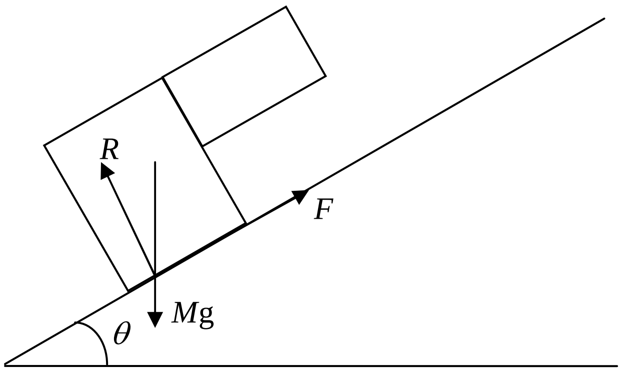
$$\bar{x} = \frac{0.24\rho \times 0.2 + 0.08\rho \times 0.6}{0.24\rho + 0.08\rho} = 0.3\,\text{m}$$ **M1A1**

$$\bar{y} = \frac{0.24\rho \times 0.3 + 0.08\rho \times 0.5}{0.24\rho + 0.08\rho} = 0.35\,\text{m}$$ **M1A1**

(ii) The critical position for equilibrium is when G lies directly above P. **1**

Then $\theta = \angle GPU = \tan^{-1}\left(\frac{\bar{x}}{\bar{y}}\right) = \tan^{-1}\left(\frac{0.3}{0.35}\right) = \tan^{-1}\left(\frac{6}{7}\right) = 40.6°$ **M1A1**

(iii) Diagram **1**



At critical friction:

Resolving parallel to the plane: $\mu R = F = 0.32\rho g \sin\theta$ **1**

Resolving perpendicular to the plane: $R = 0.32\rho g \cos\theta$ **1**

$\tan\theta = \frac{F}{R} = \mu$, so $\theta = \tan^{-1}\mu = \tan^{-1}(0.9) = 42.0°$ **M1A1**

(iv) It will topple first. **1**

B3(i) The ellipse is $\frac{x^2}{10^2}+\frac{y^2}{5^2}=1$

Differentiating implicitly, $\frac{2x}{100}+\frac{2y}{25}\frac{\mathrm{d}y}{\mathrm{d}x}=0$ **1**

So $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{25x}{100y}=-\frac{25\times 6}{100\times 4}=-\frac{3}{8}$ **1**

Therefore the equation of the tangent is $y=-\frac{3}{8}x+c$ **1**

where, since the tangent must pass through $(6,4)$

$c=4+\frac{3}{8}\times 6=6\tfrac{1}{4}$

So the equation is $y=-\frac{3}{8}x+\frac{25}{4}$

or $3x+8y=50$ **1**

(ii) The tangents at the end of the major axes have equations $x=\pm 10$. **1**

On the tangent in part (ii), when $x=10$, $y=2\tfrac{1}{2}$. This is M_2. **2**

When $x=-10$, $y=10$. This is M_1. **2**

(iii) The foci are at $(\pm ae,0)$ where $b^2=a^2(1-e^2)$

So $e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{100-25}{100}}=\sqrt{0.75}=\frac{\sqrt{3}}{2}$ **1**

So $ae=5\sqrt{3}$, and $F=(5\sqrt{3},0)$ is a focus of the ellipse. **1**

(iv) The gradient of M_1F is $\frac{10-0}{-10-5\sqrt{3}}$ **1**

The gradient of M_2F is $\frac{2\frac{1}{2}-0}{10-5\sqrt{3}}$ **1**

The product of these is $\frac{25}{-100+75}=-1$ **1**

So the two lines are perpendicular. **1**

B4(i) $\dfrac{w-u}{w-v} = \dfrac{2+3i-9i}{2+3i-3-5i}$ **1**

$= \dfrac{2-6i}{-1-2i}$ **1**

$= \dfrac{(2-6i)(-1+2i)}{(-1-2i)(-1+2i)}$ **1**

$= \dfrac{-2+4i+6i+12}{1+4}$ **1**

$= 2+2i$ **1**

The argument of this is $\dfrac{\pi}{4}$ **1**

(ii) $\dfrac{z-u}{z-v} = \dfrac{x+iy-9i}{x+iy-3-5i}$

$= \dfrac{(x+i(y-9))(x-3-i(y-5))}{(x-3+i(y-5))(x-3-i(y-5))}$ **1**

$\dfrac{(x(x-3)+(y-9)(y-5))+i((y-9)(x-3)-x(y-5))}{(x-3)^2+(y-5)^2}$ **1**

Since the argument of this is $\dfrac{\pi}{4}$ the real and imaginary parts must be equal. **1**

So $x^2 - 3x + y^2 - 14y + 45 = xy - 3y - 9x + 27 - xy + 5x$ **1**

This simplifies to $x^2 + y^2 + x - 11y + 18 = 0$ **1**

(iii) Completing the squares, $\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{11}{2}\right)^2 - \frac{1}{4} - \frac{121}{4} + 18 = 0$ **1**

$\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{11}{2}\right)^2 = \frac{1}{4} + \frac{121}{4} - 18 = \dfrac{1+121-72}{4} = \dfrac{50}{4} = 12\frac{1}{2}$ **1**

This is a circle centre $\left(-\dfrac{1}{2}, \dfrac{11}{2}\right)$ and radius $\sqrt{12\frac{1}{2}} = \dfrac{5\sqrt{2}}{2}$ **2**

B5(i) $y = \dfrac{2x^2 + 11x + 5}{x^2 - 4x + 3} = \dfrac{(2x+1)(x+5)}{(x-1)(x-3)}$ **1**

The horizontal asymptotes correspond to the zeros of the denominator. **1**
They are therefore $x = 1$ and $x = 3$. **2**

$y = \dfrac{2 + \dfrac{11}{x} + \dfrac{5}{x^2}}{1 - \dfrac{4}{x} + \dfrac{3}{x^2}} \to 2$ as $x \to \pm\infty$. **1**

So $y = 2$ is the horizontal asymptote. **1**

(ii) The curve cannot meet vertical asymptotes, so consider the horizontal one. **1**

If $\dfrac{2x^2 + 11x + 5}{x^2 - 4x + 3} = 2$ then $2x^2 + 11x + 5 = 2x^2 - 8x + 6$ **1**

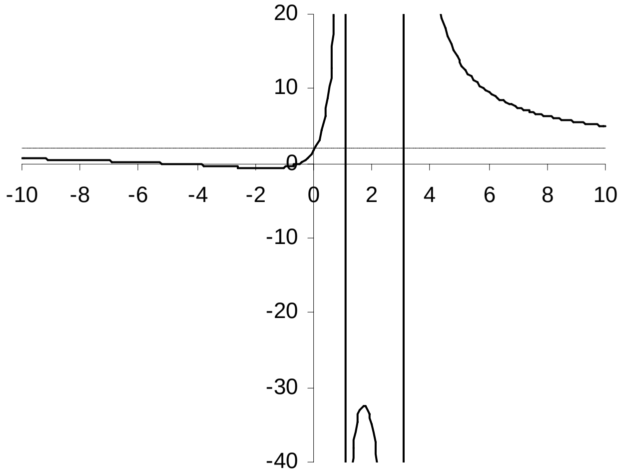
So $19x = 1$ $x = \dfrac{1}{19}$ **1**

(iii) When $x = 0$, $y = \dfrac{5}{3}$ **1**

When $y = 0$, $(2x+1)(x+5) = 0$

so $x = -\dfrac{1}{2}$ or $x = -5$ **2**

(iv) Sketch of graph showing features found above. Turning points and other features are not required. **3**



B6a	$\alpha + \beta + \gamma = -2$	**1**
	$\alpha\beta + \alpha\gamma + \beta\gamma = -3$	**1**
	$\alpha\beta\gamma = -6$	**1**
	So $\beta\gamma + \gamma\alpha + \alpha\beta = -3$	**1**
	$\beta\gamma\gamma\alpha + \beta\gamma\alpha\beta + \gamma\alpha\alpha\beta = \alpha\beta\gamma(\alpha + \beta + \gamma) = -6 \times (-2) = 12$	**1**
	$\beta\gamma\gamma\alpha\alpha\beta = (\alpha\beta\gamma)^2 = 36$	**1**
	So the required cubic equation is $x^3 + 3x^2 + 12x - 36 = 0$	**1**
b(i)	$x^2 + 12x + 11 = (x+6)^2 - 25$	**M1A1**
(ii)	$\int_0^4 \frac{dx}{\sqrt{x^2 + 12x + 11}} = \int_0^4 \frac{dx}{\sqrt{(x+6)^2 - 25}}$	
	Let $u = x + 6$. Then $\frac{du}{dx} = 1$.	**1**
	When $x = 0, u = 6$	
	When $x = 4, u = 10$	**1**
	The integral $= \int_6^{10} \frac{du}{\sqrt{u^2 - 5^2}}$	**1**
	$= \left[\cosh^{-1}\frac{u}{5}\right]_6^{10} = \cosh^{-1} 2 - \cosh^{-1}\frac{6}{5}$	**1**
	$= \ln\left(2 + \sqrt{4-1}\right) - \ln\left(\frac{6}{5} + \sqrt{\frac{36}{25} - 1}\right)$	**1**
	$= \ln\left(2 + \sqrt{3}\right) - \ln\left(\frac{6 + \sqrt{11}}{5}\right) = \ln\left(\frac{10 + 5\sqrt{3}}{6 + \sqrt{11}}\right)$	**1**