Further Mathematics Semester 2 2007 Version 1
Mark Scheme
Section A
Answer All Questions

A1 $\frac{3}{2x+5} > x$

Multiply both sides by $(2x+5)^2$

$3(2x+5) > x(2x+5)^2$ **1**

$x(2x+5)^2 - 3(2x+5) < 0$

$(2x+5)(2x^2+5x-3) < 0$

$(2x+5)(2x-1)(x+3) < 0$

$(x+3)\left(x+\frac{5}{2}\right)\left(x-\frac{1}{2}\right) < 0$ **3**

$x < -3$ or $-\frac{5}{2} < x < \frac{1}{2}$. **1**

This question may also be answered by drawing graphs.

A2 $\sum_{r=1}^{n}(r+5)(3r+4) = \sum(3r^2+19r+20)$ **1**

$= 3\sum r^2 + 19\sum r + 20\sum 1$

$= \frac{3}{6}n(n+1)(2n+1) + \frac{19}{2}n(n+1) + 20n$ **2**

$= \frac{n}{2}(2n^2+3n+1+19n+19+40)$

$= n(n^2+11n+30)$

$= n(n+5)(n+6)$. **2**

A3(i) $z = 2-3\text{i}, w = 1+4\text{i}$

$w - z = -1+7\text{i}$ **1**

(ii) $w^* = 1-4\text{i}$ **1**

(iii) $\frac{w-z}{w^*} = \frac{-1+7\text{i}}{1-4\text{i}} = \frac{(-1+7\text{i})(1+4\text{i})}{(1-4\text{i})(1+4\text{i})}$ **1**

$= \frac{-1-4\text{i}+7\text{i}-28}{1+16}$ **1**

$= \frac{-29+3\text{i}}{17}$

$= -\frac{29}{17} + \frac{3}{17}\text{i}\ (\approx -1.71 + 0.176\,\text{i})$. **1**

A4 $A=\begin{pmatrix}2 & -3\\ 4 & 1\end{pmatrix},\ B=\begin{pmatrix}3 & 1\\ 2 & 0\end{pmatrix}$

(i) $A-B=\begin{pmatrix}-1 & -4\\ 2 & 1\end{pmatrix}$ **1**

(ii) $AB=\begin{pmatrix}0 & 2\\ 14 & 4\end{pmatrix}$ **4**

A5 Horizontal component is $4-7\cos 30° = 4-\frac{7\sqrt{3}}{2} \approx -2.06$ N. **1**

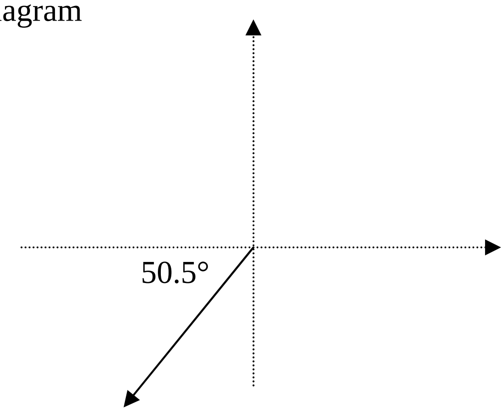
Vertical component is $-6+7\cos 60° = -\frac{5}{2} = -2.5$ N. **1**

Magnitude is $\sqrt{\left(4-\frac{7\sqrt{3}}{2}\right)^2+\left(\frac{5}{2}\right)^2}$

$=\sqrt{16+\frac{147}{4}-28\sqrt{3}+\frac{25}{4}} = \sqrt{59-28\sqrt{3}} \approx 3.24$ N. **1**

Direction = $\tan^{-1}(V/H) \approx 50.5°$ down and left. **1**

Diagram



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A6 $\int_1^6 \frac{1}{\sqrt{9+x^2}}\,dx = \left[\sinh^{-1}\left(\frac{x}{3}\right)\right]_1^6 = \sinh^{-1} 2 - \sinh^{-1}\frac{1}{3}$ **2**

$= \ln\left(2+\sqrt{5}\right) - \ln\left(\frac{1}{3}+\sqrt{\frac{1}{9}+1}\right)$

≈ 1.116 **2**

= 1.12 to 2dp. **1**

A7	The cross product of $2\mathbf{i}+\mathbf{j}-3\mathbf{k}$ and $2\mathbf{j}+5\mathbf{k}$ is $11\mathbf{i}-10\mathbf{j}+4\mathbf{k}$	**2**
	so the equation is $11x-10y+4z=d$.	**1**
	Since it passes through $\mathbf{i}-3\mathbf{j}+4\mathbf{k}$, $d=11+30+16=57$	
	So the equation is $11x-10y+4z=57$.	**1**
	Alternatively,	
	Comparing coefficients:	
	$x=1+2s$	
	$y=-3+s+2t$	
	$z=4-3s+5t$	**2**
	Eliminating t	
	$5y=-15+5s+10t$	
	$-2z=-8+6s-10t$	
	$5y-2z=-23+11s$	**1**
	Eliminating s	
	$10y-4z=-46+22s$	
	$11x=11+22s$	
	$11x-10y+4z=57$	**1**

Section B
Answer 6 questions

B1(i) The matrix required is $\begin{pmatrix} \cos 30° & \sin 30° \\ -\sin 30° & \cos 30° \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$. **3**

(ii) This is reflection in the x-axis. **2**

(iii) $A^{-1} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$. **1**

(iv) $A^{-1}BA = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$. **3**

(v) This is reflection in a line inclined at an angle of $30°$ to the horizontal. **2**

B2(i) The equation of the hyperbola is $xy = 4$. **1**

(ii) Differentiating implicitly, $y + x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{y}{x} = -\dfrac{2/t}{2t} = -\dfrac{1}{t^2}$. **1**

The equation of the tangent is $y = -\dfrac{1}{t^2}x + c$, **1**

where $c = \dfrac{2}{t} + \dfrac{1}{t^2}2t = \dfrac{4}{t}$.

That is $y = -\dfrac{1}{t^2}x + \dfrac{4}{t}$. **1**

(iii) $L = \left(0, \dfrac{4}{t}\right), M = (4t, 0)$. **2**

(iv) The gradient of the normal is t^2. **1**

The equation of the normal is $y = t^2x - 2t^3 + \dfrac{2}{t}$. **1**

At G, $y = x,\ x(1 - t^2) = -2t^3 + \dfrac{2}{t} = \dfrac{2}{t}(1 - t^4)$.

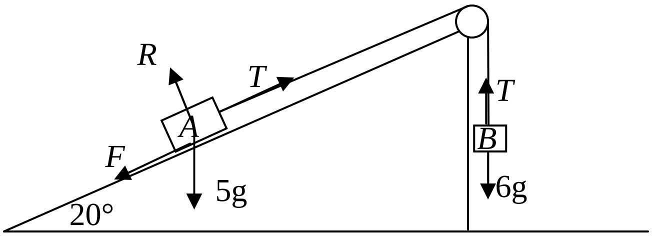
So $x = y = \dfrac{2}{t}(1 + t^2)$. Therefore $G = \left(\dfrac{2}{t}(1 + t^2), \dfrac{2}{t}(1 + t^2)\right)$. **1**

(v) The gradient of LG is $\dfrac{\frac{2}{t}(-1 + t^2)}{\frac{2}{t}(1 + t^2)} = \dfrac{-1 + t^2}{1 + t^2}$.

The gradient of MG is $\dfrac{\frac{2}{t}(1 + t^2)}{\frac{2}{t}(1 - t^2)} = \dfrac{1 + t^2}{1 - t^2}$. **1**

The product of these is –1 so the lines are perpendicular. **1**

B3(i)



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(ii) Resolving perpendicular to the plane, $R = 5g\cos 20° = 46.04$.

Friction force, $F = \mu R = 9.209 \approx 9.21\,\text{N}$. **1**

(iii) Resolving parallel to the plane, $5a = T - F - 5g\cos 70° = T - 25.97$. **2**

Forces on body B: $6a = 6g - T = 58.8 - T$. **1**

Adding: $11a = 58.8 - 25.968 = 32.83$.

Therefore $a = 2.985 \approx 2.98\,\text{ms}^{-2}$. **1**

(iv) Distance travelled in 2 seconds from rest is

$s = ut + \frac{1}{2}at^2 = 0 + 2a = 5.97$ m. (Accept 5.96.) **1**

(v) New acceleration (deceleration) $d = -\frac{25.97}{5} = -5.194$. **1**

Maximum velocity $= u + at = 0 + 2.985 \times 2 = 5.969$. **1**

Distance to stop $= \frac{v^2 - u^2}{2d} = \frac{0 - 5.969^2}{2 \times -5.194} = 3.43\,\text{m}$. **1**

B4(a)(i)

$$\cosh x \cosh y + \sinh x \sinh y = \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$$

$$= \frac{e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y} + e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y}}{4}$$ **1**

$$= \frac{e^{x+y} + e^{-x-y}}{2} = \cosh(x+y).$$ **1**

(ii) $\cosh 3x = \cosh(x+2x) = \cosh x \cosh 2x + \sinh x \sinh 2x$ **1**

$= \cosh x(\cosh^2 x + \sinh^2 x) + \sinh x(2\sinh x \cosh x)$ **1**

$= \cosh^3 x + 3\cosh x \sinh^2 x = \cosh^3 x + 3\cosh x(\cosh^2 x - 1)$

$= 4\cosh^3 x - 3\cosh x$. **1**

(b)(i) $f(x) = \tan\left(\frac{\pi}{4} + x\right)$, $f(0) = \tan\frac{\pi}{4} = 1$

$f'(x) = \sec^2\left(\frac{\pi}{4} + x\right)$, $f'(0) = \sec^2\frac{\pi}{4} = 2$ **1**

$f''(x) = 2\sec\left(\frac{\pi}{4} + x\right)\sec\left(\frac{\pi}{4} + x\right)\tan\left(\frac{\pi}{4} + x\right)$,

$f''(0) = 2\sec^2\frac{\pi}{4}\tan\frac{\pi}{4} = 4$ **1**

$f'''(0) = 4\sec^2\frac{\pi}{4}\tan^2\frac{\pi}{4} + 2\sec^4\frac{\pi}{4} = 16$ **1**

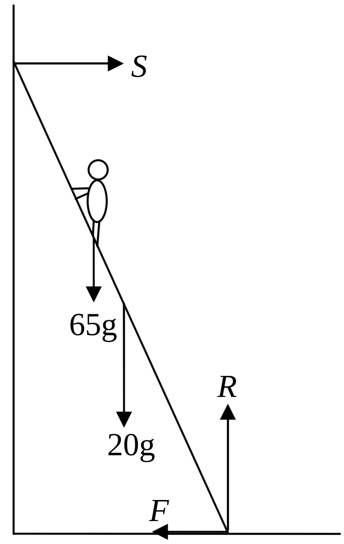
Expansion required is $1 + 2x + \frac{4x^2}{2!} + \frac{16x^3}{3!} = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \ldots$ **1**

(ii) $\tan(0.8) = \tan\left(\frac{\pi}{4} + 0.0146018\right)$

$\approx 1 + 2 \times 0.0146018 + 2 \times 0.0146018^2 + \frac{8}{3} \times 0.0146018^3$ **1**

≈ 1.029638 to 6dp. **1**

B5(i) Diagram



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(ii) Resolving forces vertically, $R = (20+65)g = 85g$ N. **1**

(iii) Resolving forces horizontally, $S = F < \mu R = 0.3R = 25.5g$ N. **2**

(iv) Let x m be the distance of the man up the ladder at the moment when it slips.
Taking moments, $20g4\cos 70° + 65gx\cos 70° = 25.5g8\sin 70°$. **2**

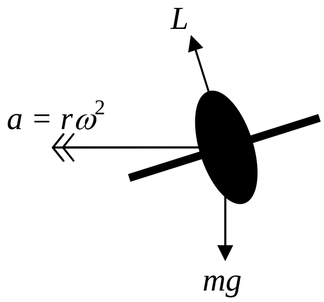
So $x = \dfrac{204\sin 70° - 80\cos 70°}{65\cos 70°} = 7.39$ m **2**

B6(i) It has $5 \times 60 = 300$ seconds to complete a full circle, which is 2π radians.
So angular velocity is $\omega = \dfrac{2\pi}{300} = 0.02094 \approx 0.0209$ radians per second. **M1A1**

(ii) The speed of the plane is 500 kmh^{-1},
which is $500 \times \dfrac{10}{36} = 138.9 \approx 139 \text{ ms}^{-1}$. **M1A1**

(iii) $v = r\omega$, so $r = \dfrac{v}{\omega} = \dfrac{138.9}{0.02094} = 6630$ metres or 6.63 km.
(Accept 6650m or 6.65 km.) **M1A1**

(iv) diagram



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resolving forces vertically: $mg = L\cos\theta$. **1**

resolving horizontally: $mr\omega^2 = L\sin\theta$. **1**

So $\tan\theta = \dfrac{r\omega^2}{g} = 0.2968$. (Accept 0.2975.)

Then $\theta = 16.5°$ (Accept $\theta = 16.6°$.) **1**

B7(i) $\frac{z-(3+2\mathrm{i})}{z-(4+5\mathrm{i})} = \frac{7+4\mathrm{i}-3-2\mathrm{i}}{7+4\mathrm{i}-4-5\mathrm{i}} = \frac{4+2\mathrm{i}}{3-\mathrm{i}}$ **1**

$= \frac{(4+2\mathrm{i})(3+\mathrm{i})}{(3-\mathrm{i})(3+\mathrm{i})} = \frac{12+4\mathrm{i}+6\mathrm{i}-2}{9+1} = \frac{10+10\mathrm{i}}{10} = 1+\mathrm{i}$. **M1A1**

(ii) The complex number is in the first quadrant and $\tan^{-1}\theta = 1$, so $\theta = \frac{\pi}{4}$. **1**

(iii) $\frac{z-(3+2\mathrm{i})}{z-(4+5\mathrm{i})} = \frac{x+y\mathrm{i}-3-2\mathrm{i}}{x+y\mathrm{i}-4-5\mathrm{i}} = \frac{x-3+(y-2)\mathrm{i}}{x-4+(y-5)\mathrm{i}}$

$= \frac{(x-3+(y-2)\mathrm{i})(x-4-(y-5)\mathrm{i})}{(x-4+(y-5)\mathrm{i})(x-4-(y-5)\mathrm{i})}$ **1**

$= \frac{(x-3)(x-4)-(x-3)(y-5)\mathrm{i}+(y-2)(x-4)\mathrm{i}+(y-2)(y-5)}{(x-4)^2+(y-5)^2}$ **1**

If the argument of this complex number is $\frac{\pi}{4}$

then the real part must equal the imaginary part.

Therefore $(x-3)(x-4)+(y-2)(y-5) = -(x-3)(y-5)+(y-2)(x-4)$ **1**

That is $x^2-7x+12+y^2-7y+10 = -xy+5x+3y-15+xy-4y-2x+8$

So $x^2+y^2-10x-6y+29=0$, as required. **1**

(iv) The equation can be written as $(x-5)^2+(y-3)^2=5$. **1**

This is a circle with centre $(5, 3)$ and radius $\sqrt{5}$. **2**

B8(i) The parametric equation of l is $\mathbf{r} = 7\mathbf{i}-3\mathbf{j}+11\mathbf{k}+\lambda(3\mathbf{i}-2\mathbf{j}+5\mathbf{k})$. **1**

(ii) Substituting this into the equation of the plane:

$3(7+3\lambda)-2(-3-2\lambda)+5(11+5\lambda)=6$

$21+9\lambda+6+4\lambda+55+25\lambda=6$

$38\lambda+76=0,\ \lambda=-2$. **1**

So B is the point with position vector $\mathbf{r} = \mathbf{i}+\mathbf{j}+\mathbf{k}$. **1**

(iii) $\overrightarrow{AB} = -6\mathbf{i}+4\mathbf{j}-10\mathbf{k}$, **1**

with length $AB = \sqrt{6^2+4^2+10^2} = \sqrt{152}\ (\approx 12.3)$. **1**

(iv) $C = 2\mathbf{i},\ \overrightarrow{AC} = -5\mathbf{i}+3\mathbf{j}-11\mathbf{k}$ **1**

$D = -3\mathbf{j},\ \overrightarrow{AD} = -7\mathbf{i}-11\mathbf{k}$ **1**

The volume of the tetrahedron is $\frac{1}{6}\left|\overrightarrow{AB}\cdot\overrightarrow{AC}\times\overrightarrow{AD}\right|$ **1**

$= \frac{1}{6}\left|(-6\mathbf{i}+4\mathbf{j}-10\mathbf{k})\cdot(-5\mathbf{i}+3\mathbf{j}-11\mathbf{k})\times(-7\mathbf{i}-11\mathbf{k})\right|$ **1**

$= \frac{1}{6}\left|(-6\mathbf{i}+4\mathbf{j}-10\mathbf{k})\cdot(-33\mathbf{i}+22\mathbf{j}+21\mathbf{k})\right|$ **1**

$= \frac{1}{6}\times 76 = \frac{38}{3}\ (\approx 12.7)$. **1**

B9(a) $\frac{dy}{dx} + 2xy = 2x$

The integrating factor is $e^{\int 2x dx} = e^{x^2}$. **1**

Multiplying by this: $e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = 2xe^{x^2}$.

The left hand side is exact: $\frac{d}{dx}\left(e^{x^2}y\right) = 2xe^{x^2}$. **1**

Integrating: $e^{x^2}y = e^{x^2} + c$. **1**

Finally: $y = 1 + ce^{-x^2}$. **1**

(b)(i) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 5t + 9$

The auxiliary equation is $m^2 + 4m + 5 = 0$, $m = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$. **1**

The complementary function is $Ae^{-2t}\cos t + Be^{-2t}\sin t$. **1**

For a particular integral try $y = Ct + D$, $\frac{dy}{dt} = C$, $\frac{d^2y}{dt^2} = 0$. **1**

Substituting in the equation: $0 + 4C + 5Ct + 5D = 5t + 9$.
Equating coefficients: $5C = 5$, $4C + 5D = 9$.

Solving: $C = 1$, $D = 1$. **1**

General solution: $y = Ae^{-2t}\cos t + Be^{-2t}\sin t + t + 1$. **1**

(ii) Differentiating:

$\frac{dy}{dt} = -2Ae^{-2t}\cos t - Ae^{-2t}\sin t - 2Be^{-2t}\sin t + Be^{-2t}\cos t + 1$.

When $t = 0$, $1 = A + 1$, $3 = -2A + B + 1$, $A = 0$, $B = 2$. **1**

Final solution: $y = 2e^{-2t}\sin t + t + 1$. **1**