Mark Scheme Further Mathematics July 2006 Version 1

Section A
Answer all questions

A1 Given that are the roots of the equation $x^2-5x-2=0$ are α and β ;

(i) $(\alpha+\beta)$ = 5, $\alpha\beta$ = -2 **[2]**

(ii) $\alpha^2\beta+\alpha\beta^2$ = $\alpha\beta(\alpha+\beta)$ = -10 **[2]**

(iii) Find a quadratic equation which has roots $\alpha^2\beta+\alpha\beta^2$ and α^3 and β^3. **1**

$\alpha^3\beta^3$ = -8

$x^2+10x-8=0$ **1**

A2 Solve the inequality $\dfrac{3x}{x-1}>x$.

Sketch the curves, asymptotes x=1 and y=3 OR multiply both sides by $(x-1)^2$ **2**

Points of intersection: $x^2 - 4x = 0$ **1**

x=0 or x=4 **2**

From the sketch: x<0 or 1 < x < 4 (5 marks for answers by any method)

A3

$$\sum_{r=1}^{n}\left(r+\frac{1}{2^r}\right) = \sum_{r=1}^{n}(r)+\sum_{r=1}^{n}\frac{1}{2^r}) \quad \mathbf{1}$$

$$= \frac{n(n+1)}{2}+(G.P.\,first\,term=1/2, ratio=1/2) \quad \mathbf{1}$$

$$= \frac{n(n+1)}{2}+\frac{\frac{1}{2}(1-(\frac{1}{2})^n)}{\frac{1}{2}} \quad \mathbf{1}$$

$$= \frac{n(n+1)}{2}+1-\frac{1}{2^n} \quad \mathbf{1}$$

A4 A complex number α is given by $\alpha = 1+i5$.

(i) $\alpha^* = 1-i5$ **1**

(ii) $\alpha\,\alpha^*$ = 26 **1**

(iii) $\dfrac{\alpha+\alpha^*}{\alpha^*} = \dfrac{1+i5+1-i5}{1-i5}$ **1**

1

$$= \frac{2(1+i5)}{(1-i5)(1+i5)} = \frac{1}{13}+i\frac{5}{13}$$

A5 $\mathbf{r} = (t^3 - 3t)\mathbf{i} + 4t^2\mathbf{j}$, $t\geq 0$

(i) $\mathbf{v} = (3t^2 - 3)\mathbf{i} + 8t\mathbf{j}$ **2**

(ii) When P is moving parallel to the vector **i + j,** the coefficient of **i =** coeff. of **j.** **m1**

$(3t^2 - 3) = 8t$ **1**

$3t^2 - 8t - 3 = 0$

$(3t + 1)(t - 3)$

t= -1/3, 3 **1**

$t\geq 0 \;\therefore\; t=3$ **1**

A6 $u=\sin x$, $\dfrac{du}{dx}=\cos x$ **1**

$\int\cos 2x\cos x\,dx = \int(1-2\sin^2 x)\cos x\,dx$ **1**

$= \int (1-2u^2)du$ **1**

$= \quad u - \frac{2u^3}{3} + c = \sin x - \frac{2\sin^3 x}{3} + C$ **1**

A7 L.H.S.= $\cosh 2x = \frac{1}{2}(e^{2x} + e^{-2x})$ **1**

R.H.S. = $2\cosh^2 x - 1 = 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1$

$= \frac{e^{2x} + 2 + e^{-2x}}{2} - 1$ **1**

$= \frac{1}{2}(e^{2x} + e^{-2x})$ = L.H.S. **1**

A8 A has position vector 2**i** – **j** + **k** and B has position vector 3**j** + 4**j** - 2**k** .
Area = ½ |OAxOB|

OAxOB = $\begin{vmatrix} \mathbf{i} & j & k \\ 2 & -1 & 1 \\ 3 & 4 & -2 \end{vmatrix}$ **M1**

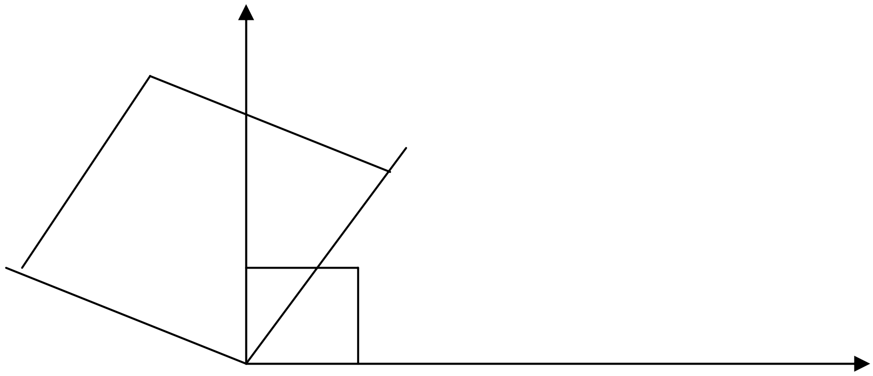
= -2**i** + 7**j** +11**k** **1**

Area = ½($\sqrt{174}$) **1**

Section B

B1

(i) $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 & -1 \\ 0 & 2 & 1 & 3 \end{pmatrix}$ calculate coordinates **2**



sketch **1**

(Both areas above should be a square) Determinant = 5 **1**

Unit square has area = 1. Transformed square has side of length $\sqrt{5}$, the area is 5, i.e. the area is 5 x area of the unit square. **2**

(ii) the angle of **R** = $\tan^{-1}2 = 63.4^{o} = \theta$ **1**

The other transformation is an enlargement, s.f. $\sqrt{5}$. **2**

(iii) **R** = $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$ **M1** **A1**

B2 A chord of a parabola from P ($at_1^2, 2at_1$) to Q ($at_2^2, 2at_2$) subtends a right angle at the vertex, i.e. angle POQ is a right angle.

(i) Gradient of OP = , $\frac{2at_1}{at_1^2} = \frac{2}{t_1}$ **2**

Gradient of OQ = , $\frac{2at_2}{at_2^2} = \frac{2}{t_2}$ **1**

OP and OQ are perpendicular **M1**

$\frac{2}{t_1} . \frac{2}{t_2}$ = -1 **A1**

$t_1 t_2 = -4$

(ii) Write down the coordinates of the mid-point of PQ

$x = \frac{a}{2}(t_1^2 + t_2^2)$, $y = a(t_1 + t_2)$ **1,1**

(iii) $(t_1^2 + t_2^2) = (t_1 + t_2)^2 - 2t_1 t_2$ **1**

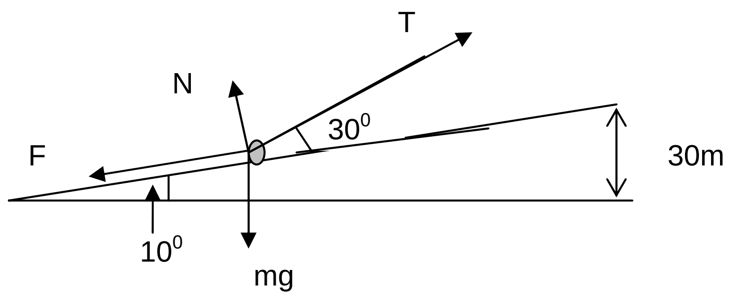
Substitute $\frac{y}{a} = t_1 + t_2$ and $t_1 t_2 = -4$ **M1**

$x = \frac{a}{2}(\frac{y^2}{a^2} + 8)$ **1**

$y^2 = 2ax - 8a^2 = 2a(x - 4a)$ **1**

B3

(i)



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(ii) Resolve forces perpendicular to the slope: **M1**

N + T sin 30^0 = mg cos 10^0 **1**

N = 6 x 9.8 cos 10^0 -20 x 0.5 = 47.9066

= 47.9 (3 s.f.) **1**

(iii) Resultant force parallel to the slope = ma **M1**

T cos 30^0 - F - mg sin 10^0 = 6 x 0.4 **1**

F = 4.71 N **1**

The coefficient of friction = 4.71/47.9 **1**

μ = 0.098

(iv) s = ut + ½ at^2 **M1**

200 = 0 + ½ (0.4) t^2 **1**

t = 31.6 sec (3 s.f.) **1**

B4

(i)

Change in momentum of A: 0.24 x 8 – 0.24 x 6 = 0.48 Ns **M1**
Magnitude of the impulse = 0.48 N s. **A1**

(ii) Conservation of Momentum: **M1**

8 x 0.24 = 6 x 0.24 + mv

$v \geq 6$ **A1**

$0.48/m \geq 6$ **1**

$m \leq 0.48/6 = 0.08$

(iii) Find the coefficient of restitution between A and B

$$e = \frac{speed\ of\ separation}{speed\ of\ approach} = \frac{v-6}{8}$$ **M1**

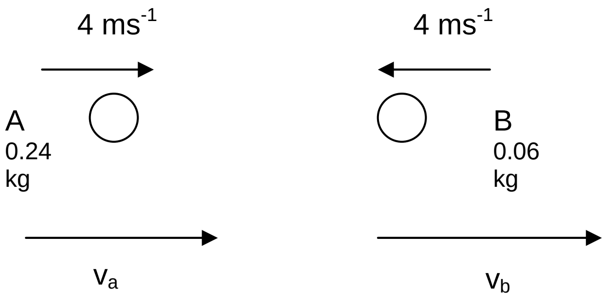
Impulse on B = Change in Mom of B **1**

0.48 N s = 0.06 v

$v = 8\ m\ s^{-1}$ **1**

e = ¼

On another occasion A and B are travelling towards each other, each with a speed of 4 m s^{-1}, when they collide directly.



(iv)

Conservation of momentum:

$$0.24 \times 4 - 0.06 \times 4 = 0.24\ v_a + 0.06\ v_b$$

Newton's Law of restitution: **1**

$$\tfrac{1}{4} = \frac{v_b - v_a}{4 - -4}$$

$$2 = v_b - v_a$$

Solving the equations simultaneously gives

$v_a = 2,\ v_b = 4ms^{-1}$ **1**

1

B5
(a)

$$(x-1)^2 = (\sin\theta - \cos\theta)^2$$
$$= 1 - 2\sin\theta\cos\theta$$
$$= 1 - y$$
$$y = 1 - x^2 + 2x - 1$$
$$= -x(x-2)$$

3

or substitute

$$-x(x-2) = -(\sin\theta - \cos\theta + 1)\ (\sin\theta - \cos\theta - 1)$$
$$= -\left(\left(\sin\theta - \cos\theta\right)^2 - 1^2\right)$$
$$= -(1 - 2\sin\theta\cos\theta - 1)$$
$$= y$$

(b) (i) $2\sinh^2 x - 5\cosh x = 10$

$2\cosh^2 x - 5\cosh x - 12 = 0$ **1**

$(2\cosh x + 3)(\cosh x - 4) = 0$ **1**

$\cosh x \neq -3/2 \quad \cosh x = 4$ **1**

$x = \ln(4 + \sqrt{16-1}) = \ln(4 + \sqrt{15})$ **1**

(ii) $f'(x) = 4\sinh x\cosh x - 5\sinh x$ **1**

Stationary values: $4\sinh x\cosh x - 5\sinh x = 0$

$\sinh x(4\cosh x - 5) = 0$

$\sinh x = 0 \quad or \quad \cosh x = 5/4$ **1**

$x = 0,\ \ln(\tfrac{5}{4} + \sqrt{\tfrac{25}{16} - 1})$

$x = 0,\ \ln 2$ **2**

B6 (a) Tension in the horizontal string : T_1

Tension in the second string : T_2

Angle made by second string with the horizontal, θ = $\cos^{-1}(24/25) = 16.26^o$

(i) Resolving forces vertically (no motion): $T_2 \sin\theta = mg$ **M1**

$T_2 = 0.08 \times 9.8 / 0.280 = 2.8$ N **A1**

Horizontally: Resultant force = m.acceleration

$T_1 + T_2\cos\theta = 0.08 \times 10.5^2/2.4$ **M2**

$T_1 = 0.987$ N **A1**

(ii) Lower string is taut if $T_1 >= 0$

$0.08 \times v^2/2.4 \geq T_2\cos\theta$ **M1**

Least value of v: v= 8.98 **A1**

(b) Period: $\frac{2\pi}{\omega} = 0.2$ **1**

$\omega = 10\pi$ **1**

Max speed: $v = a\omega = 0.3 \times 10\pi = 3\pi$ (= 9.42) ms^{-1} **1**

Max acceleration: acc = $\omega^2 a = 100\pi^2 \times 0.3 = 30\pi^2$ = (296) m s^{-2} **1**

B7 (a) $z^n = \cos n\theta + i\sin n\theta$ **1**

$\frac{1}{z^n} = z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$ **1**

$z^n + \frac{1}{z^n} = 2\cos n\theta$ **1**

(b)(i) $\left(z - \frac{1}{z}\right)^4 = z^4 + 4z^3(\frac{-1}{z}) + 6z^2(\frac{-1}{z})^2 + 4z(\frac{-1}{z})^3 + (\frac{-1}{z})^4$ **1**

$= z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}$ **1**

(ii) $\left(z - \frac{1}{z}\right)^4 = (\cos\theta + i\sin\theta - \cos\theta + i\sin\theta)^4 = 16\sin^4\theta$ **1**

1

$= 2\cos 4\theta - 8\cos 2\theta + 6$ **1**

$8\sin^4\theta = \cos 4\theta - 4\cos 2\theta + 3$ **1**

(c) $\cos 4\theta + 1 = \cos 4\theta - 4\cos 2\theta + 3$

$4\cos 2\theta = 2$

$\cos 2\theta = \frac{1}{2}$

$2\theta = \pm\frac{\pi}{3}, \pm\frac{5\pi}{3`}.$

$\theta = \pm\frac{\pi}{6}, \frac{5\pi}{6}$ **M1**

A1

B8

(a) $\overrightarrow{AB} = \begin{bmatrix} -5 \\ -3 \\ 3 \end{bmatrix}$, $\overrightarrow{AC} = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}$ **1**

(i) $\overrightarrow{AB}$ x $\overrightarrow{AC}$ = $\begin{bmatrix} 12 \\ -16 \\ 4 \end{bmatrix}$ **1**

(ii) the exact value of the area of triangle ABC = ½(4)$\sqrt{9+16+1}$=2$\sqrt{26}$ **2**

(iii) the Cartesian equation of the plane Π containing A, B and C.

12x- 16y+ 4z=c

Substitute A (3,4,1)

c = 36 – 64 + 4 = -24 **1**

3x – 4y + z = -6 **2**

(b) The line l passes through the point D (0, -5, 0) and is perpendicular to plane Π.
Find:

(i) the equation of l in the form $\mathbf{r} = \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$; **2**

(ii) the coordinates of the point of intersection of l with Π.

Substitute from l in Π: 3(3λ) – 4(-5 -4λ) + λ = -6

26 λ = -26 **1**

λ = -1 **1**

the point of intersection of l with Π; (-3, -1, -1)

B9

a) Integrating Factor $e^{\int 3dx} = e^{3x}$ — Can also be done using an Auxiliary equation etc **1**

$ye^{3x} = \int 5e^{-4x}e^{3x}dx$ **1**

$= 5\int e^{-xdx}$ **1**

$= -5e^{-x} + C$ **1**

$y = 5e^{-4x} + Ce^{-3x}$ **1**

1

(b) Auxiliary equation: $m^2 + 2m - 3 = 0$

$(m-1)(m+3) = 0$

Complementary Function: $y = Ae^{x} + Be^{-3x}$ **1**

Particular Integral $y = Ce^{-4x}$ **1**

$\frac{dy}{dx} = -4Ce^{-4x}$, $\frac{d^2y}{dx^2} = 16Ce^{-4x}$

$16Ce^{-4x} + 2(-4Ce^{-4x}) - 3Ce^{-4x} = 5e^{-4x}$

C = 1 **1**

P.I. $y = e^{-4x}$

General Solution: $y = Ae^{x} + Be^{-3x} + e^{-4x}$ **1**

$y = 3$ when $x = 0$. 3 = A + B +1

$\frac{dy}{dx} = 0$ when $x = 0$. $\frac{dy}{dx} = Ae^{x} - 3Be^{-3x} - 4e^{-4x}$ **1**

0 = A – 3B -4

A+B=2

A-3B = 4, A = 5/2, B = -1/2

$y = \frac{5}{2}e^{x} - \frac{1}{2}e^{-3x} + e^{-4x}$ **1**