Fractional Brownian Motion for Stock Price and Call Price Modelling

Abstract

In order to forecast future stock prices and option prices, models have traditionally been constructed with the random element determined by Brownian motion. It's known that Brownian motion is part of the generalised form named fractional Brownian motion (fBm) which has a varying Hurst parameter between 0 and 1. Brownian motion has a H value equal to 0.5. In this dissertation we explore the background theory to provide an improved stock price and call option price model. We have shown that alternative H values can produce better predictions for the stock price and for the call option price. Also when $H = 0.5$ in the Fractional Brownian motion model the stock price proved to be better than the original geometric Brownian motion (gBm) model even though theoretically they should have been the same. For a particular call option we achieve a method for finding the optimum H value to minimise the error.

1 Introduction

At the present time there is a growing amount of research on the theory of applying fractional Brownian motion (fBm) to the stock market. However there seems to be a lack of data explicitly supporting the theory showing that fractional Brownian motion has better results than the traditional model Brownian motion to predict future stock prices and call option prices. The observed stock data have been taken from Yahoo finance and the options data downloaded from the internet site optionsdata.net. If the results show that fBm stock price model is better for speculating on the stock market than Brownian motion stock price models, then a trader would prefer to use this as a tool in order to go long or short on a particular stock. Also derivative companies who offer options would be advised to use the fBm Black Scholes equation to determine the price of an option to safeguard the company from a loss and allow them to make a profit in the options' business.

2 The Role of Brownian Motion for Stock Price Modelling

2.1 Properties of Brownian Motion

Brownian motion is a random process that depends continuously on $t \in [0, T]$ and satisfies the following conditions [24]:

- $B(0) = 0$.
- For $0 \leq s \leq t \leq T$ the random variable given by the increment $B(t) - B(s)$ is normally distributed with mean zero and variance t-s. Equivalently $B(t) - B(s) \sim \sqrt{t-s}N(0,1)$.
- For $0 \leq s < t < u < v \leq T$ the increments $B(t) - B(s)$and $B(v) - B(u)$ are independent.
- For computational purpose we discretize Brownian motion where $\delta t = \frac{T}{N}$.
- The derivative of Brownian motion $\frac{dB(t)}{dt}$ is used as input noise in dynamical systems.
- Self similarity which means invariance in distribution under a suitable change of scale.
- Continuity
- Normality - the increments $B_{s+t} - B_s$ have a normal distribution $N(0,t)$
- Markov property - the conditional distribution B_t given information up to time s where $s < t$ depends only on $B(s)$

2.2 Problems of Brownian Motion as a Model of Stock Price

- When on its own there is a probability $= \frac{1}{2}$ that Brownian motion is negative. However the stock price is never negative and has some growth element. Therefore the model requires a positive deterministic function of time as well.

2.3 Geometric Brownian Motion Model of the Stock Price

Brownian motion on its own would lead to the stock price having negative values. Stock prices are always greater than zero so it's written as an exponential which can only be positive. We will define the differential which contains two parts: the deterministic part to permit trend in the stock price and the stochastic elements to allow fluctuations of the price.
$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$

Determining the parameters μ and σ

$dS_t = S_t\mu dt + S_t\sigma dB_t$
Using Ito's representation the solution is
$ln\frac{S_{t+1}}{S_t} = [\mu - \frac{\sigma^2}{2}]dt + \sigma dB_t$

We will define $\hat{\mu}$ as the mean of the natural logarithm of the stock price rate change.
$\hat{\mu} = E[ln\frac{S_{i+1}}{S_i}]$
$E[ln\frac{S_{t+1}}{S_t}] = E[(\mu - \frac{\sigma^2}{2})dt] + E[\sigma dB_t]$
$E[ln\frac{S_{t+1}}{S_t}] = E[(\mu - \frac{\sigma^2}{2})dt]$
$\hat{u} = E[\mu - \frac{\sigma^2}{2}]dt$
$\hat{u} = E[\mu - \frac{\sigma^2}{2}]\tau$
$\mu = \frac{\hat{u}}{\tau} + \frac{\sigma^2}{2}$
$[ln\frac{S_{t+1}}{S_t}]^2 = [\mu - \frac{\sigma^2}{2}]^2 dt^2 + 2[\mu - \frac{\sigma^2}{2}]dt\sigma dB_t + [\sigma dB_t]^2$
$E[(ln\frac{S_{t+1}}{S_t})^2] = E[[\mu - \frac{\sigma^2}{2}]^2 dt^2] + \sigma^2 t$

We will define v^2 as the volatility of the natural logarithm of the stock price rate change.
$v^2 = E[(ln\frac{S_{t+1}}{S_t})^2] - E[ln\frac{S_{t+1}}{S_t}]^2$
$v^2 = \sigma^2 t$
Standard deviation $=\sigma = \frac{v}{\sqrt{t}}$

3 Fractional Brownian Motion

Brownian motion belongs to a family called fractional Brownian motion (fBm) with varying Hurst values in the interval $[0, 1]$. The traditional stock price models have been based on Brownian motion $B(t)$ with $H = \frac{1}{2}$. It's important to discuss whether this is the optimum fBm to model the stock price. We will firstly list the main properties of fBm.

- Gaussian process $t \in R$
- $E_{\mu_\phi}[B_H(t)] = 0$
- For simplicity assume $B_H(0) = 0$
- $E_{\mu_\phi}[B_H(t)B_H(s)] = \frac{1}{2}[t^{2H} + s^{2H} - (t-s)^{2H}]$
- $Var_{\mu_\phi}(B_H(t)) = E_{\mu_\phi}[B_H(t)^2] = t^{2H}$
- For $H \in (0, 1)$ fractional Brownian motion is self similar in the sense that $B_H(\alpha t)$ has the same laws as $B_H(t)$

3.1 Correlation of Fractional Brownian Motion

Fractional Brownian motion can be divided into three groups depending on the nature of its correlation:

3.1.1 No Correlation fBm Process

for $H = \frac{1}{2}$
The observations are uncorrelated.
This is practical if the trader is a follower of the Efficient Market Hypothesis believing that increments are independent.

3.1.2 Long Memory fBm Process

for $\frac{1}{2} < H < 1$
$\rho(k) > 0$ for k large enough.
$\sum_{-\infty}^{\infty} |\rho(k)| = \infty$
The process X_i has long memory and long range dependence.
This long range dependence is a property that makes it useful for stock price modelling. Some traders would argue that stocks moves in trends which conflict with the Efficient Market Hypothesis idea that prices are unpredictable.

3.1.3 Negative Correlated fBm Process

for $0 < H < \frac{1}{2}$
$\rho(k) < 0$ for k large enough
$\sum_{-\infty}^{\infty} |\rho(k)| < \infty$
The process is also said to have short-range dependence.
Short-range dependence is a measure of the decline in statistical dependence of two events separated by successively longer spans of time.

3.2 Drawback of fBm as a Stock Price Model - not a Semi-Martingale

If an investor has a portfolio of shares and bonds and is guaranteed not to make a loss, therefore we have an arbitrage situation. It's natural for the stock market to aim to eliminate giving away guaranteed profits so the model shouldn't allow it either. To be free from arbitrage a stock price model's stochastic part must have the martingale property. Fractional Brownian motion is only a martingale at $H = \frac{1}{2}$. For $H \neq \frac{1}{2}$ fractional Brownian motion isn't a semi martingale for the following reasons:

- $H < \frac{1}{2}$ the quadratic variation for B^H doesn't exist so therefore can't be a semi-martingale.
- for $H > \frac{1}{2}$ the process will be in the following form $B^H_t = B^H_0 + M + A$. Suppose B^H is a semi martingale therefore $[M, M]_t = [B^H, B^H]_t = 0$. This implies $B^H_t = B_0 = 0$ a.s for all t. Hence $B^H_t = A$ so has finite variation which leads to a contradiction.

The fact that fBm lacks the martingale property is a major drawback which creates the impression of arbitrage opportunities in speculating in the stock market. This is against the stock market principle that there is 'no such thing as a free lunch.' When trading on the stock market there is always a possibility of losing, therefore the model shouldn't say otherwise.

3.3 Integration with respect to Fractional Brownian Motion

We will aim to use stochastic calculus to integrate a function with respect to fBm.
Regarding integration with respect to fBm there are two types: pathwise integration and Wick Ito integration. In the following section we will discuss both types and explore the relationship between them.

3.3.1 Pathwise Integration

Let $\pi : a = t_0 < t_1 < t_2, \cdots < t_n = b$ be a partition of $[0, T]$
$\int_0^T f(t, \omega) dB^H_s = \lim_{|\Delta| \mapsto 0} \sum_{i=0}^{n-1} f(t_k, \omega)[B^H(t_{i+1}) - B^H(t_i)]$
$|\Delta| = max_{0 \leq k \leq n-1}(t_{k+1} - t_k)$
These integrals don't have expectation zero.

3.3.2 Wick Ito Skorokhod Integration

In order for fBm to be arbitrage free we need the martingale property
$E[X(t)|F_s] = X(s)$
$X(t) = X(s) + \int_s^t f(u) dB^H_u$
Applying expectations to both sides
$E[X(t)] = E[X(s)] + E[\int_s^t f(u, \omega) dB^H_u]$
Unfortunately in the case when f(u) is a stochastic function $E[\int_s^t f(u, \omega) dB^H_u] \neq 0$
For $X(t)$ to be a martingale we need a different operator to ensure the integral becomes zero.
The Wick integral product is defined as:
$\int_0^T f(t, \omega) \diamond dB^H_s = \lim_{|\Delta| \mapsto 0} \sum_{i=0}^{n-1} f(t_k, \omega) \diamond [B^H(t_{i+1}) - B^H(t_i)]$
In contrast to pathwise integration
$E[\int_0^T f(t, \omega) \diamond dB^H_s] = 0$

We define $F(\omega)$ and $G(\omega)$ [30] .
$F(\omega) = \sum_{\alpha \in I} a_\alpha H_\alpha(\omega)$
$G(\omega) = \sum_{\beta \in I} a_\alpha H_\beta(\omega)$

- I is the set of all multi indices
- $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m) \in I$
- $H_\alpha(\omega) = h_{\alpha_1}((\omega, e_1))h_{\alpha_2}((\omega, e_2)) \cdots h_{\alpha_m}((\omega, e_m))$

Consequently the Wick product of $F(\omega)$ and $G(\omega)$ is the following:
$(F \diamond G)(\omega) = \sum_{\alpha,\beta \in I} a_\alpha b_\beta H_{\alpha+\beta}(\omega)$

In this case we will let $f, g \in L^2_\phi(R)$. This implies that f,g are deterministic functions it can be shown that [24]

$$(\int_R f dB_H) \diamond (\int_R g dB_H) = (\int_R f dB_H)(\int_R g dB_H) - (f, g)_\phi \tag{1}$$

Applying expectations to both sides we note two main properties

- $E[(\int_R f dB_H) \diamond (\int_R g dB_H)] = 0$
- For deterministic functions
  $E[(\int_R f dB_H)(\int_R g dB_H)] = E[(\int_R f \diamond dB_H)(\int_R g \diamond dB_H) = (f, g)_\phi$

3.3.3 Relation between the Wick Integral and Pathwise Integral with respect to Fractional Brownian Motion

The relation between the Wick integral and pathwise integral is the following [20]
$\int_0^T F(B_H(t))dB_H(t) = \int_0^T F(B_H(t)) \diamond dB_H(t) + H \int_0^T F'(B_H(t))t^{2H-1}dt$

We wish to find a condition for the pathwise integral and Wick Ito integral to be the same.
$E[\int_0^T F(B_H(t)) \diamond dB^H_t] = 0$
$E[\int_0^T F(B^H_t)dB^H_t] = E[H \int_0^T F'(B^H_t)t^{2H-1}dt]$
When $F' = 0$ ie when the function is deterministic.
$\int_0^T F(B^H_t) \diamond dB^H_t = \int_0^T F(B^H_t)dB^H_t$

If function F is deterministic ie a non random function we can ensure a linear process $X(T)$ which is in the form
$X(T) = X(0) + \int_0^T F(B^H_t)dB^H_t$
can become a martingale.

3.3.4 Properties of Wick Ito Integration

Let us denote the space L^2_ϕ
$\phi : RxR \mapsto R$
$\phi(s, t) = H(2H - 1)|t - s|^{2H-1}$
For Borel measure functions f,g $[0, T] \mapsto R$
$||f||^2_\phi = \int_0^T \int_0^T f(s)f(t)\phi(s, t)dsdt$
$(f, g)_\phi = \int_0^T \int_0^T f(s)g(t)\phi(s, t)dsdt$
For a deterministic function $f \in L^2_\phi([0, T])$
f_n be the step function approximating f
$f_n = \sum_i a_k \chi_{[t_k,t_{k+1})}(t) \mapsto f(t)$
$\int_0^T f_n(t) \diamond dB^H_t = \sum a_k[B^H_{t_{k+1}} - B^H_{t_k}]$

For the continuous case when $n \mapsto \infty$
$\int_0^T f(t) \diamond dB^H_t = \lim_{n\mapsto\infty} \int_0^T f_n(t) \diamond dB^H_t$

We will now list the main properties of Wick Ito integration.

- $E[\int_0^T f(t) \diamond dB_t^H] = 0$

- $E[\int_0^T f(t) \diamond dB_t^H \int_0^T g(t) \diamond dB_t^H] = (f, g)_\phi$

- $E[(\int_0^T f(t) \diamond dB_t^H)^2] = ||f||_\phi^2$

3.4 Geometric Fractional Brownian Motion

Similar to geometric Brownian motion the stock price differential has a deterministic component and a stochastic part. However we use instead fractional Brownian motion to drive the random component.
$dS_t = \mu S_t dt + \sigma S_t dB_H(t)$
$\mu \in R$, $\sigma > 1$ and $H \in (\frac{1}{2}, 1)$

The solution to the differential equation is the following:
$S_t = S_0 exp(\mu t + \sigma B_H(t))$

A stock price model should have a separate trend part and noise part. The latter part should just be noise. It shouldn't dictate the direction of the stock price. The general trend is calculated by μ, the average of the change of stock price. A stochastic process that has an unpredictable nature is called a martingale.

However fractional Brownian motion doesn't have the martingale property, but has long term memory for $H > \frac{1}{2}$
$f(B_H(t)) = f(B_H(0)) + \int_0^t f'(B_H(0))dt$
$E[f(B_H(t))] \neq f(B_H(0))$

To remove arbitrage we need to use the Wick product which allows the expectation of the stochastic component to be zero. Therefore the stochastic differential equations becomes
$dS_t = \mu S_t dt + \sigma S_t \diamond dB_H(t)$

Wick Exponential

Before we solve a stochastic process involving a Wick product, we need to know how to interpret an exponential containing the Wick product.
Firstly let us define $X^{\diamond n}$.
$X^{\diamond n} = X \diamond X \diamond \cdots \diamond X$ (n factors)
$exp \diamond (X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^{\diamond n}$
$B_H(t) = < w, \chi_{[0,t]}(.) > = (\chi_{[0,t]}(.), e_k)_\phi < \omega, e_k >$
It can be proven that [24]
$exp \diamond (\sigma B_H(t)) = exp(\sigma B_H(t) - \frac{\sigma^2 t^H}{2})$

Using Wick Integrals to solve the Fractional Stochastic Equations

Let the differential of a stochastic process S_t contain the Wick product.
$dS_t = \mu S_t dt + \sigma S_t \diamond dB_H(t)$
$S(0) = S_0$
$\frac{dS_t}{dt} = \mu S_t + \sigma S_t \diamond W_H(t)$
$\frac{dS_t}{dt} = [\mu + \sigma W_H(t)] \diamond S_t$
$dS_t = [\mu t + \sigma \int_0^t W_H(s) ds] \diamond S_t$

Integrating both sides.
$S_t = S_0 exp^\diamond(\mu t + \sigma \int_0^t W_H(s) ds)$
$S_t = S_0 exp^\diamond(\mu t + \sigma B_H(t))$

The Wick product is for stochastic proccesses so we can factor out the deterministic part
$S_t = S_0 exp(\mu t) exp \diamond (\sigma B_H(t))$
$S_t = S_0 exp(\mu t - \frac{\sigma^2 t^{2H}}{2} + \sigma B_H(t))$
$E_{\mu_\phi}[S_t] = S_0 exp(\mu t)$

4 Fractional Black Schole's Equation

The original Black Schole's equation is a means to calculate the fair option price. It eliminates arbitrage opportunities for both the buyer and the seller of the call option. The equation is based on the assumption that the stock price follows a geometric

Brownian motion with constant drift and volality. Perhaps this assumption is wrong and so we aim to enlarge the scope and assume the stock price follows a geometric fractional Brownian motion. Let $F(S,T)$ be the call option price where T is the duration of the period, S_T is the stock price at the end of the period, K is the strike price, ρ the bank interest rate and σ is volatility of the stock.
$F(S,T) = S_0\phi(d_1) + Kexp(-\rho T)\phi(d_1 - \sigma T^H)$
$d_1 = \frac{ln\frac{S_T}{K}+[\rho+\frac{\sigma^2}{2}]T}{\sigma T^H}$
$\phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} exp(-\frac{s^2}{2})ds$
Evidently when $H = \frac{1}{2}$ we have the traditional Black Schole's equation. For this equation there is now an additional unknown value H as well as σ. In the experiments below we will choose the values that will minimise the error between the predictions and observed values.

4.1 Arbitrage opportunities with Fractional Brownian Motion

We are going to show that pathwise integration with respect to fBm will create arbitrage opportunities for the trader. [22]

Example

Let us define a portfolio $V^\theta(t)$ composed of a quantity of stocks $S_0(t)$ and bonds $S_1(t)$ and for simplicity let $\sigma = 1$.
$V^\theta(t) = \phi_0(t)S_0(t) + \phi_1(t)S_1(t)$
$\phi_0(t) = 1 - exp(2B_H(t))$
$\phi_1(t) = 2(exp(B_H(t)) - 1)$
$dS_0(t) = rS_0(t)dt$
$dS_1(t) = S_1(t)\mu dt + S_1(t)\sigma dB_H(t)$
$S_1(t) = exp(\mu t + \sigma B_H(t))$

$V^\theta(t) = [1 - exp(B_H(t))]exp(rt) + 2exp(B_H(t) - 1)exp(B_H(t) + rt)$
$V^\theta(t) = [exp((B_H(t)) - 1]^2 exp(rt) > 0$ for $a.a < t, \omega >$
$P(V^\theta(t) \geq 0) = 1$

Therefore the portfolio is predicted using the fBm model to have guaranteed gains on the stock markets. In the real world it's rare to have guaranteed gains when speculating on the stock model except for illegal situations such as price fixing or inside trading.

4.2 Girsanov Theorem applied to Fractional Brownian Motion

B_t^H is fractional Brownian motion under the measure P.
We will define $\tilde{B}_t^H$ on the σ algebra F_t^H under the new measure Q
$\tilde{B}_t^H = B_t^H + \int_0^t \gamma_s ds$
Let K be the solution of the integral equation
$\int_R K(s)\phi(t,s)ds = \gamma(t)$ then

$\tilde{B}_t^H$ is a fractional Brownian motion under the probability measure Q on (Ω, F_t^H), which is equivalent with P.
The Radon-Nikodym derivative which relates both measures is the following [29]:

$$\frac{dQ}{dP} = exp(-\int_R K(t) \diamond dB^H(t) - \frac{1}{2}|K|_\phi^2) = exp \diamond (- < \omega, K >) \tag{2}$$

It can be shown that [29]
$E[\frac{dQ}{dP}] = 1$
This implies that we can use $\tilde{B_t^H}$ on the measure Q.

4.3 Arbitrage Free Portfolio Trading Strategy $\theta(t) = \theta(t,\omega) = (u(t), v(t))$

Here we are going to apply a trading strategy using Wick integrals to show that this portfolio is free from arbitrage
This is a $\mathscr{F}_t^H$ adapted 2 dimensional process giving the number of units u(t),v(t) held at time t of the bond and the stock.
The corresponding value process
$Z^\theta(t,\omega) = u(t)A(t) + v(t) \diamond X(t)$

Self-Financing Portfolio

$dZ^\theta(t,\omega) = u(t)dA(t) + v(t) \diamond dX(t)$
For the stock price X(t)
$dX(t) = \mu X(t)dt + \sigma X(t) \diamond dB_H(t)$

$dZ^{\theta}(t,\omega) = u(t)dA(t) + v(t)\mu \diamond X(t)dt + \sigma v(t) \diamond X(t) \diamond dB_H(t)$ where $t \in [0,T]$

For the bank account or bond A(t)
$dA(t) = \rho A(t)dt$

It can be shown that [31]
$exp(-\rho t)Z^{\theta}(t,\omega) = Z^{\theta}(0,\omega) + \int_0^t [exp(-\rho s)\sigma v(s) \diamond X(s)] \diamond d\tilde{B}_H(s)$
Let us define $f(t,\omega) = exp(-\rho s)\sigma v(s) \diamond X(s)$

Admissible Portfolio

Portfolio is admissible if :

- It is self-financing so value changes are only due to market price fluctuations.
- The stochastic process $f(t,\omega)$ belongs to the space $\tilde{L}_{\phi}^{1,2}(R)$ of all $\mathscr{F}_t^H$ adapted processes.

A stochastic process $f(t,\omega) = f(t)$ belongs to the space $\tilde{L}_{\phi}^{1,2}(R)$ if:
$\underline{E_{\tilde{\mu}_{\phi}}[\int_R f(t,\omega)) \diamond d\tilde{B}_H(t)] = 0}$

$\underline{E_{\tilde{\mu}_{\phi}}[(\int_R f(t,\omega) \diamond d\tilde{B}_H(t))^2] = ||f||^2_{\tilde{L}_{\phi}^{1,2}(R)}}$
$||f||^2_{\tilde{L}_{\phi}^{1,2}(R)} = E_{\tilde{\mu}_{\phi}}[(\int_R f d\tilde{B}_t^H)^2] = E_{\tilde{\mu}_{\phi}}[(\int_R f \diamond d\tilde{B}_t^H)^2] + E_{\tilde{\mu}_{\phi}}[(\int_R \int_R Df\phi(s,t)dsdt)^2]$

It's worth noting that if $f(t,\omega)$ is deterministic.
$||f||^2_{\tilde{L}_{\phi}^{1,2}(R)} = \int_R \int_R f(s)f(t)\phi(s,t)dsdt$

Arbitrage Portfolio

An admissible portfolio θ is called an arbitrage for this market $(A(t), X(t))$ for $t \in [0,T]$ if
$Z^{\theta}(0) \leq 0$, $Z^{\theta}(T) \leq 0$ a.s and
$\mu_{\theta}[(\omega; Z^{\theta}(T,\omega) > 0] > 0$
We deduce that no arbitrage exists by taking expectations with respect to the risk neutral probability measure $\tilde{\mu}_{\phi}$
$E_{\tilde{\mu_{\phi}}}[exp(-\rho t)Z^{\theta}(t,\omega)] = Z^{\theta}(0,\omega)$

5 Stock Price Modelling

5.1 Choice of Simulator for Fractional Brownian Motion

Davies and Harte method will be chosen since it performs well in the test and is easy to compute. A reason for this is that the simulator doesn't require storing past data unlike the Hosking and Cholesky methods and can perform the algorithms more speedily. Other dissertations such as Tom Dieker [10] who have explored a wide range of simulators and approximate simulators also comment on the pace and accuracy of this simulator.

5.2 Comparing Errors between Fractional Models and Geometric Brownian Motion Stock Price Model

Now we aim to see if fractional models are better than the geometric Brownian motion. Geometric Brownian motion has traditionally been used to model the stock price. If we find that fBm is better, not only should we consider the mean and variance of natural logarithm of the price change (section 4.7) but also the Hurst value of the stock. In this experiment errors at each day of a stock price model for the first 60 days are recorded and the norm (first,second and infinite) is computed. For a stock price model a sample of a hundred norm errors is recorded and the mean is compared with other stock price models. I accepted normality on the Kolmogorov Smirnov test with significance level 0.05 and for the t pair test accepted the alternative hypothesis that models were different with significance level 0.05. Please see results in the appendix (15.1).

1. For 2007 only the infinite norms satisfied the normality condition. H=0.8 was the best model followed by H=0.7. However the t pair test showed no distinction between the fractional models. Fractional H=0.5 proved to be a better model than gBm.

2. For 2008 only the infinite norms satisfied the normality condition. H=0.7 was the best model followed by H=0.5. However the t pair test showed no distinction between these two models. Fractional H=0.5 proved to be a better model than gBm.

3. For 2009 only the infinite norms satisfied the normality condition. H=0.8 was the best model followed by H=0.5. However the t pair test showed no distinction between these two models. Fractional H=0.5 proved to be a better model than gBm.

4. For 2010 only the infinite norms satisfied the normality condition. H=0.8 was the best model followed by H=0.7. However the t pair test showed no distinction between these two models. There was not enough evidence to show that fractional H=0.5 is a better model than gBm.

5. For 2011 only the infinite norms satisfied the normality condition. H=0.7 was the best model followed by H=0.5. However the t pair test showed no distinction between the fractional models. Fractional H=0.5 proved to be a better model than gBm.

6. For 2012 only the infinite norms for all models satisfy the normality condition. H=0.5 was the best model followed by H=0.7. However the t pair test showed no distinction between the fractional models. Fractional H=0.5 proved to be a better model than gBm.

Overall we see that fractional Brownian models perform better than geometric Brownian motion. Between the years 2007 and 2011 the assumption that $H = 0.5$ is incorrect as it leads to greater errors in the model. Throughout this period the stock shown to have an H value of 0.7 or 0.8. This isn't precise enough for a trader so he would be advised to experiment with the different H values in that range to find the minimum error. Interestingly for 2012, when $H = 0.5$ is the optimum Hurst value theoretically it shouldn't matter if the fBm model or gBm are used as they are both the same. However the fractional Brownian motion model performed better. This discrepancy will be discussed in section 12.

5.3 Predicting Future Values from Models based on Past Data

In order to reflect the real world of trading I compared future estimates from past data models in the short term (ie 11 days) for predicting the 12th day and long term ie (61 days) for predicting the 62nd day. With each model I carried out 100 simulations and calculated a sample of errors. In the short term the geometric Brownian motion formula was better than any fractional model created. However in the long term the results are considerably different. For both short and long term I compared fractional at 0.5, 0.7, 0.8 with gBm. I accepted normality on the Kolmogorov Smirnov test with significance level 0.05 and for the t pair test accepted the alternative hypothesis that models were different with significance level 0.05. Please see results in the appendix (15.2).

Short Term Prediction Results

1. For 2007 gBm was best with error of 0.0170 compared to second best H=0.8 with error 0.0209. The results showed that there wasn't significant evidence at 0.05 significance level to show that there was difference between the fractional models
2. For 2008 gBm was the best with error of 0.0938 followed by H=0.8 with error 0.1332. The results showed that there wasn't significant evidence at 0.05 significance level to show that there was difference between all the models
3. For 2009 gBm was the best with error of 0.1022 followed by H=0.8 with error 0.1526. The results showed that there wasn't significant evidence at 0.05 significance level to show that there was difference between the fractional models
4. For 2010 gBm was the best with error of 0.0539 followed by H=0.5 with error 0.0659. The results showed that there wasn't significant evidence at 0.05 significance level to show that there was difference between all the models
5. For 2011 gBm was the best with error of 0.0468 followed by H=0.7 with error 0.0706. The results showed that there wasn't significant evidence at 0.05 significance level to show that there was difference between H=0.7 and third place H=0.8.
6. For 2012 gBm was the best with error of 0.0678 followed by H=0.7 with error 0.0876. The results showed that there wasn't significant evidence at 0.05 significance level to show that there was difference between the fractional models.

Long Term Prediction Results

1. For 2007 H=0.5 was the best with error 0.0082 and the other fractional models followed closely behind compared to gBm 0.0717. There was no evidence to show distinction between the fractional models.
2. For 2008 H=0.8 was the best with error 0.0559 compared to last place gBm 0.2149. According to the t test there wasn't significant evidence at 0.05 significance level to show that there was difference between the fractional models.
3. For 2009 H=0.5 was the best model followed by H=0.7. However the t pair test showed no distinction between these two models. Fractional H proved to be a better model than gBm.

4. For 2010 H=0.7 had the least error 0.0269 with gBm in last position with error .2154. The results showed that there wasn't significant evidence at 0.05 significance level to show that there was difference between H=0.5 and second place H=0.7. However there is difference between H=0.5 and H=0.8.
5. For 2011 H=0.5 was a best model with error 0.0225 and gBm in last position with error .1013. The results showed that there wasn't significant evidence at 0.05 significance level to show that there was difference between the fractional models
6. For 2012 the fractional models were clearly better than gBm. H=0.7 was the best with error 0.0282 with others closely following. Geometric Brownian motion in last position had an error of .1094. However according to the t test there wasn't sufficient evidence to say that there is any distinction between the fractional models.

Models based on long period of data such as 61 days performed better than the short term 11 days. In the long term the model has received more information about economic conditions, so has increased confidence in the stock and therefore can produce a better estimation of future values. A trader would be advised to use past data over a wide period but that would require a significant amount of computation. For that reasons a fast simulator of fBm would be preferred.

6 Analysis of Discrepancy between GBM and Fractional Brownian Motion at H=0.5

In this section we are looking for a reason why there is a difference between the errors of gBm model and the fractional Brownian model at $H = 0.5$. Theoretically there should be no difference since they are the same model but the fractional version tends to have better stock price predictions. The error could be caused by the simulator or due to how MATLAB computes random numbers.
Let us firstly use Beran's test statistic
$T_N(H) = \frac{A_N(H)}{B_N(H)^2} = 0.2680$
$P(TN > 0.2565) \sim (1 - 0.0570) = 0.9430$
This implies that simulation generated by MATLAB program does fit a Brownian motion path.

However the Brownian motion simulator's p statistic is not as well inside the acceptance region compared to the Davies and Harte simulator with a p statistic of 0.9948.Therefore we can conclude that it's better to use the fractional simulators to model Brownian motion at $H = 0.5$ than the original stock price model. A possible explanation for this is that the simulator creates a fractional Brownian motion that shares its characteristics with the covariance matrix. The gBm doesn't enforce this property but just relies on the quality of MATLAB's random simulator.

7 Option Price Modelling

The method I used (supported by mathematicians Corrado and Su) [7] is to find the fraction of observation values outside of the bid/ask spread on the first day of each week. The sigma value used to derive the theoretical price is the value that gives the least error at $H = 0.5$ for the prior day's observations and ensures the majority of theoretical prices are within the spread. After noting the average deviation an alternative H value is used to see if the error has become even smaller. For this experiment Option 'A' weekly option price has been analysed starting from January 3rd 2013. In this next experiment I have recorded the sigma value and Hurst parameter to 2 decimal places. Results support the hypothesis that H=0.5 isn't the optimal value to minimise the average deviation between midpoint of spread and theoretical price. In general optimum Hurst values ranged between 0.36 to 0.47. In financial application for striving to get the theoretical price as close to observed prices as possible, the option pricer needs to find a minimum Hurst value and sigma value for the least error for the prior day. These parameters will be used to calculate the call price for the following day.

7.1 Finding the Stationary Points for Option A in the First Week 2013

Figure 4 shows the minimum average absolute error for different values of H and sigma. In order to computationally find this value we must find the minimum element of the average absolute error matrix - array of values influenced by sigma and H. Using the index of the minimum value and knowing the step size we find the H=0.47 and sigma = 0.215 (see appendix 14.4 on MATLAB code). Substituting these values into the next day's prediction of the call option price we see that the average deviation from the midpoint is 0.2695. This is a significant improvement compared to the original Black Scholes that estimates in the prior day sigma=0.2094 for the minimum error and has an average deviation the next day of 0.2956. This builds the case that financial derivative companies should be using the fractional Black Scholes model for determining the price of call options.

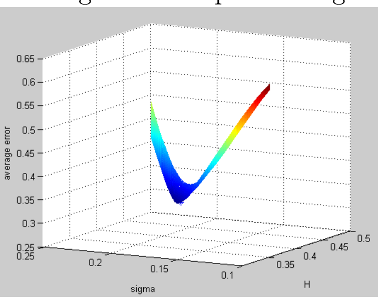
8 Conclusions and Future Work

For stock price and call option predictions it can be seen that for different H values the error has reduced. Fractional Brownian Motion has been removed of arbitrage by the Wick product but the difficulty is to explain its economic significance. The traditional geometric Brownian model is easier to understand as it ensures that the stock price is always positive and that there

Figure 1: Comparing the deviations between prediction and true value for $H = 0.5$ and the Hurst value that gives minimum error denoted as "Minhurst". The implied volatility is the sigma value that gives the minimum error for the prior day at $H = 0.5$. Evidently the error is improved by choosing an alternative H value.

Week	H=0.5 implied volatility	Hurst for minimum	H=0.5 fraction outside spread	Minhurstfractionoutsidespread	H=0.5 average deviation	Min hurst average deviation
1	.2200000	.4700000	.5470588	.4764706	.2724000	.2684000
2	.1700000	.3900000	.6058824	.6000000	.3892000	.3759000
3	.2100000	.4600000	.6781609	.5689655	.2672000	.2744000
4	.1900000	.4600000	.5328947	.4473684	.2506000	.2426000
5	.1600000	.3600000	.6625000	.6062500	.4256000	.3909000
6	.2000000	.4500000	.6062500	.5562500	.3758000	.3626000
7	.1600000	.3700000	.5375000	.5312500	.4390000	.4124000
8	.1600000	.4300000	.5285714	.5071429	.3579000	.3454000
9	.1700000	.4300000	.5642857	.5285714	.3072000	.2959000
10	.1600000	.4500000	.6071429	.5642857	.4121000	.4036000

Figure 2: Graph showing the minimum H value and sigma value to ensure the prior day's deviation is at its smallest.



are two parts - a random component and a deterministic part that sets the trend. However the advantage of the Wick product needs to be considered as without it fBm models would contain arbitrage.

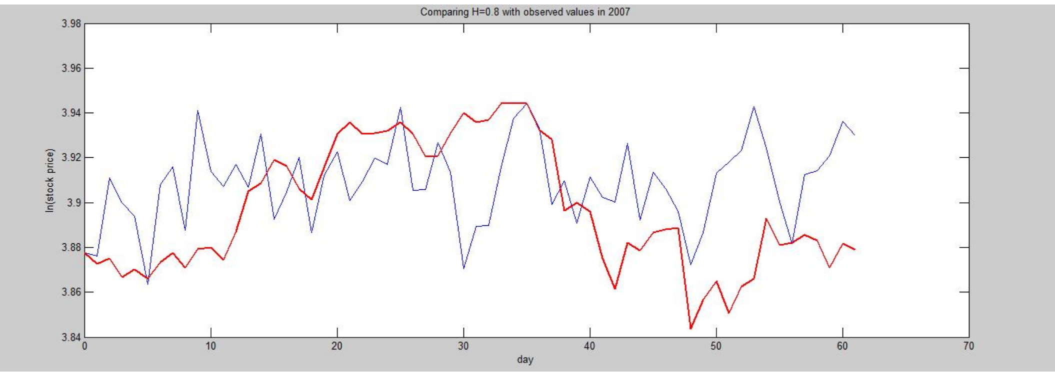
An idea to improve accuracy of predictions is to consider multiple fractional Brownian motion. Perhaps the random part of the model as just a single component simplifies the model. It's known that a stock price is not just influenced by one factor but a multiple of factors such as the global economy, the success of a company and investors selling shares to liquify their profits etc. Multiple fractional Brownian motion (MFM) is a continuous Gaussian process whose pointwise Holder exponent can be prescribed and evolves with time t. For fBm the pointwise Holder exponent and Hurst value are equal and constant. The next stage would be to make the Hurst value a function of time. Assuming that H is constant could be an over simplification since evidently stock price behaviour changes in time. For example one moment oil may be the investment of the year because of the growing need of the commodity but next moment there is volatility due to the uncertainty of a coming war in the Middle East. For the next step we will of course have to show that these multiple fractional Brownian motion models are arbitrage free before we can carry out the experiments. If the multiple fractional Brownian motion model can be proved to be more accurate, there is no reason why we should stop there. In other work for improving the call price prediction there is the Kurtosis and Skewness Adjusted Black Schole's model. In the past the natural logarithm of stock price change is considered to be a normal distribution, but in reality the tails are fatter which imply that extreme price movements are more likely, The model allows for changing the kurtosis and the skewness to reflect better the motion of the stock price. Mathematicians such as Corrado and Su [7] have shown that this model is effective. Although this model is still based on Brownian motion, it leads us to the question as to whether the results would be even better when fractional Brownian motion was incorporated into the equation.

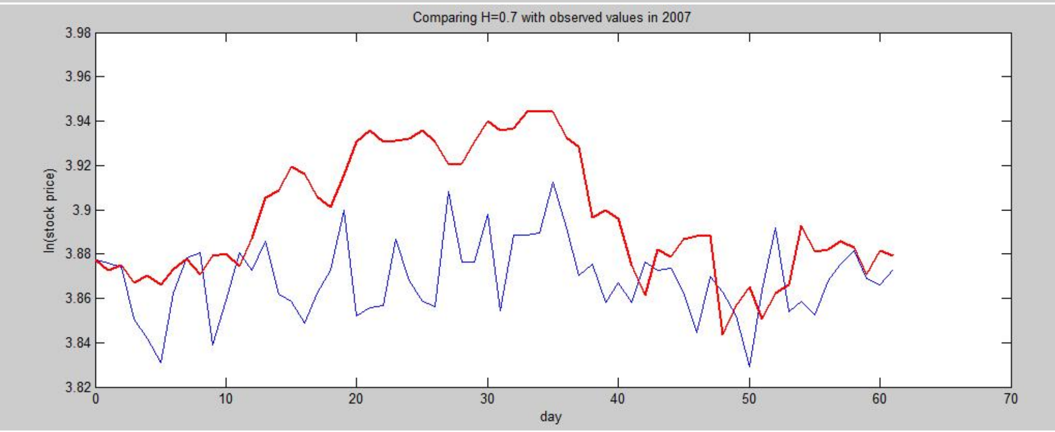
9 Appendix

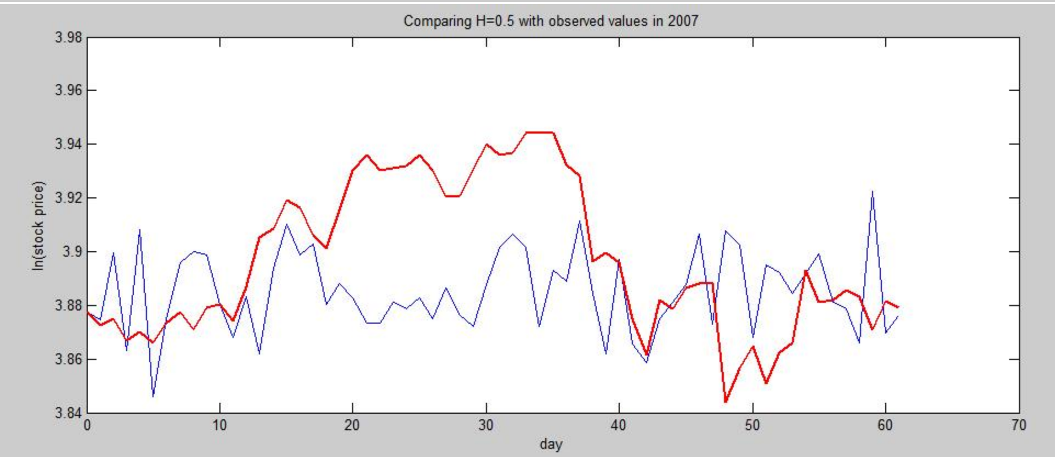
9.1 Graphs Comparing the Average of all the Models with the Observed Price

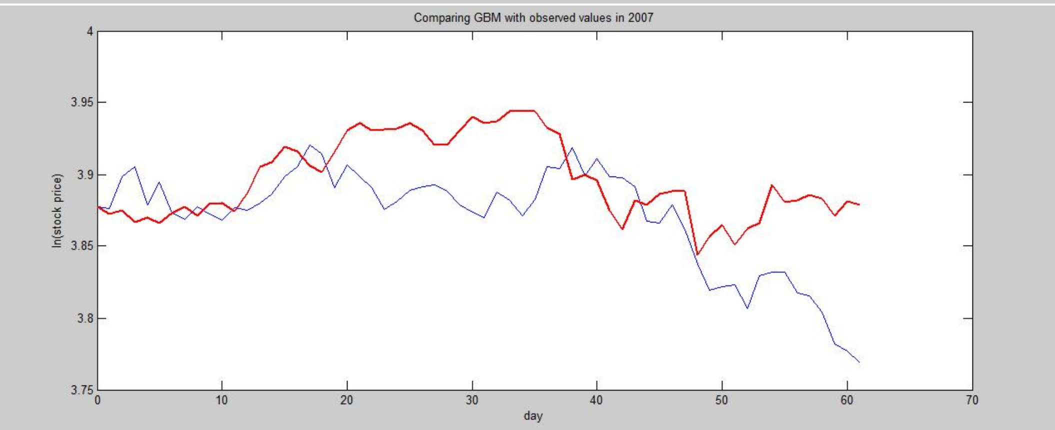
Here are the graphs for years 2007 to 2012 comparing the average model (blue line) with the observed prices (red line). The first graph for each year is the best model.

2007

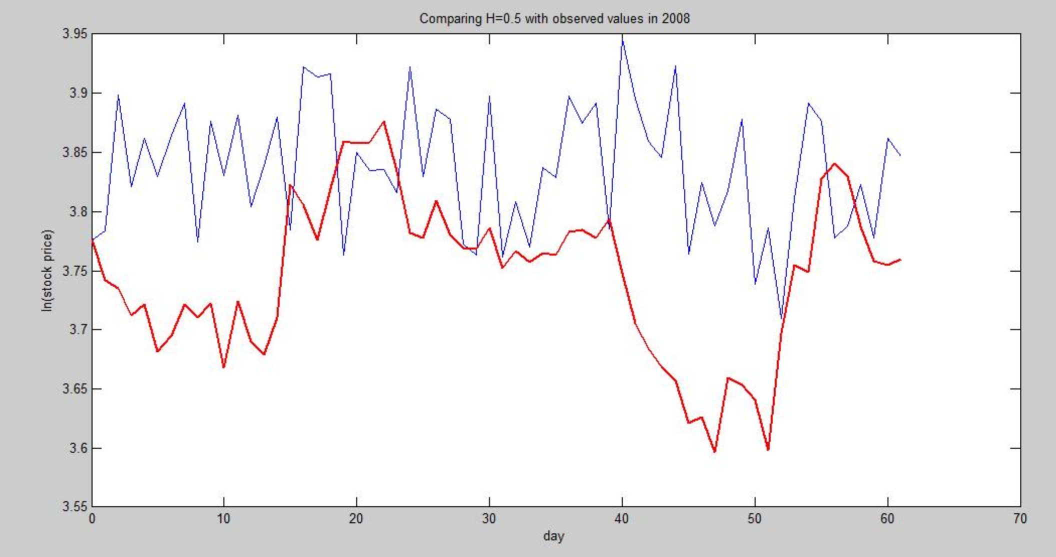




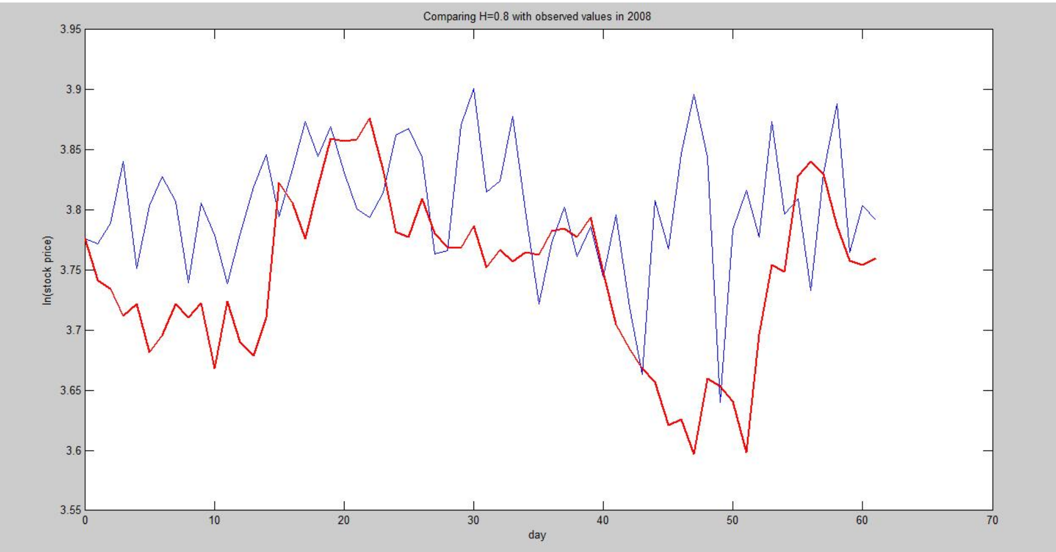




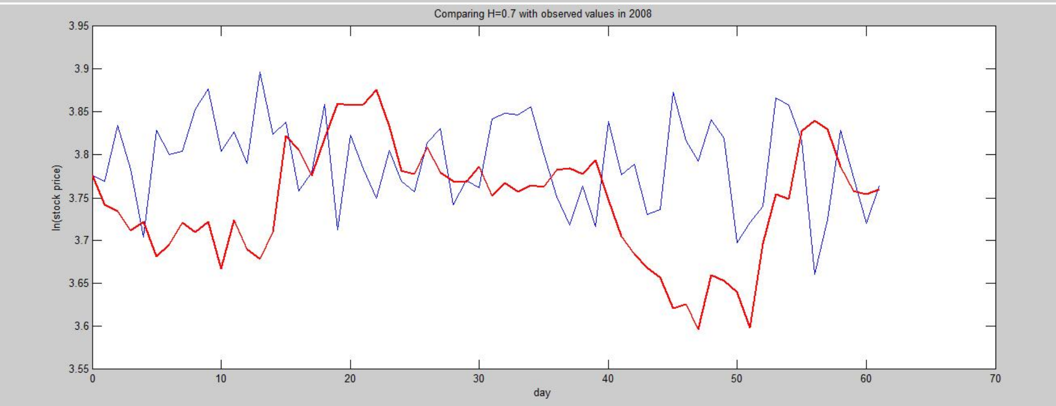
2008



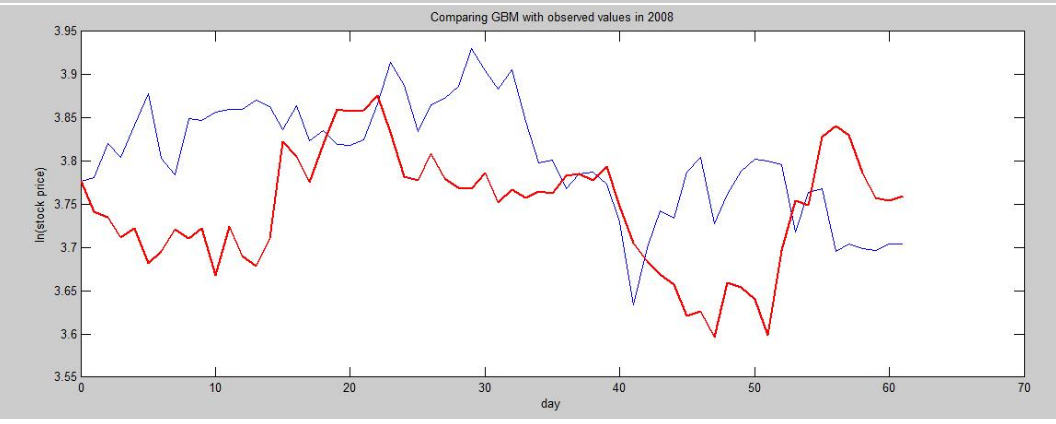
Comparing H=0.5 with observed values in 2008
ln(stock price)
day



Comparing H=0.8 with observed values in 2008
ln(stock price)
day

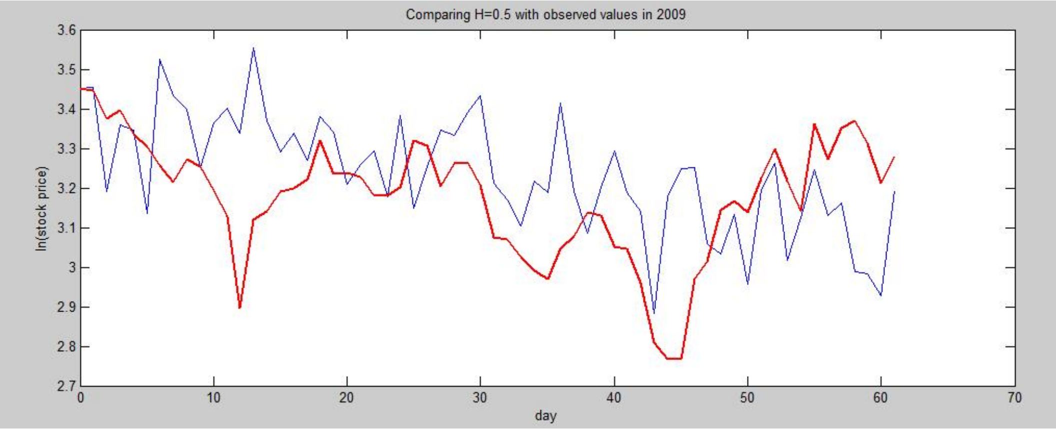


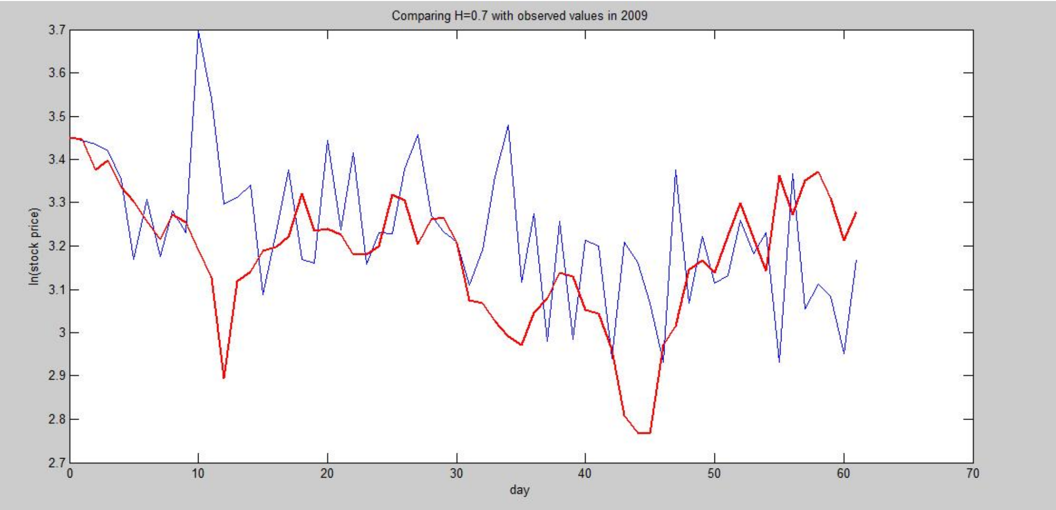
Comparing H=0.7 with observed values in 2008
ln(stock price)
day

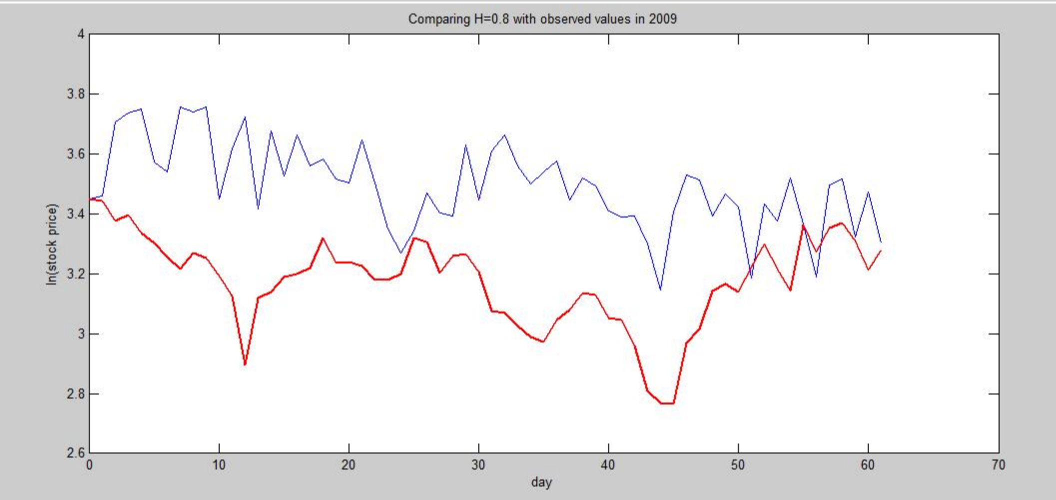


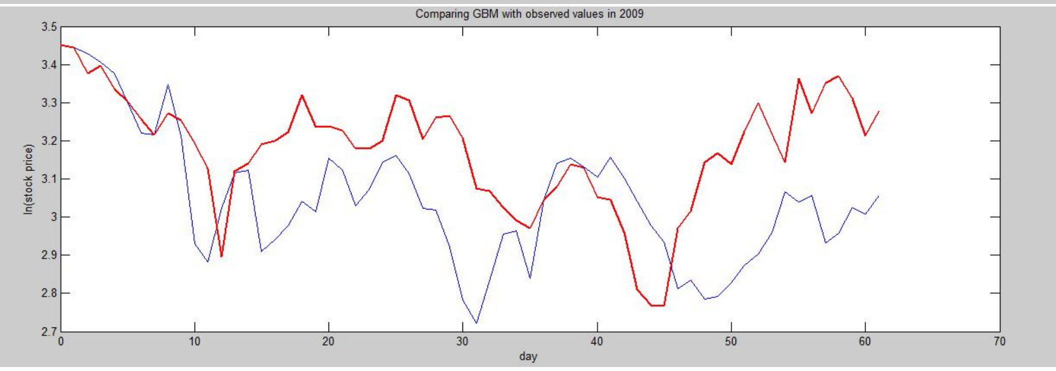
Comparing GBM with observed values in 2008
ln(stock price)
day

2009

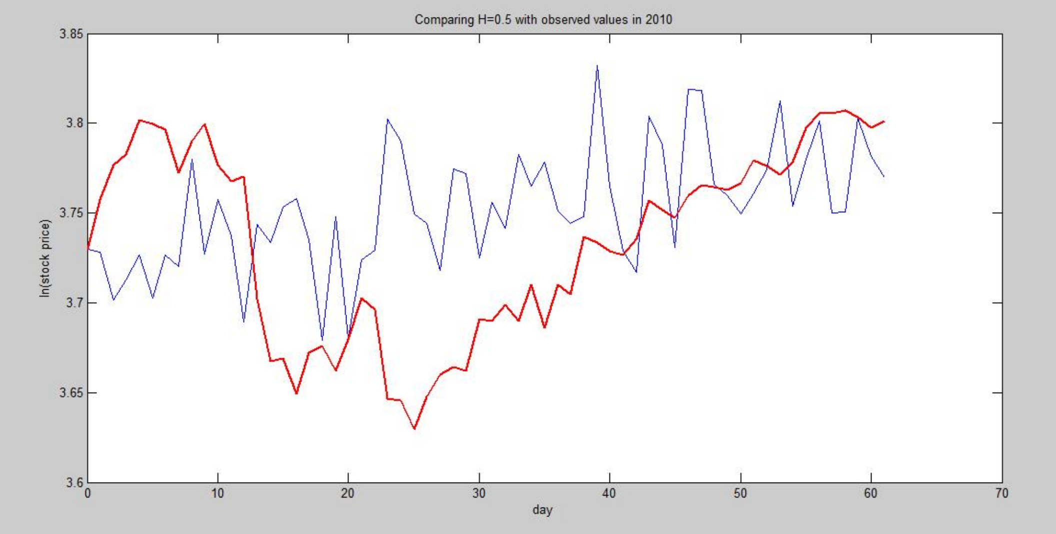

Comparing H=0.5 with observed values in 2009
ln(stock price)
day

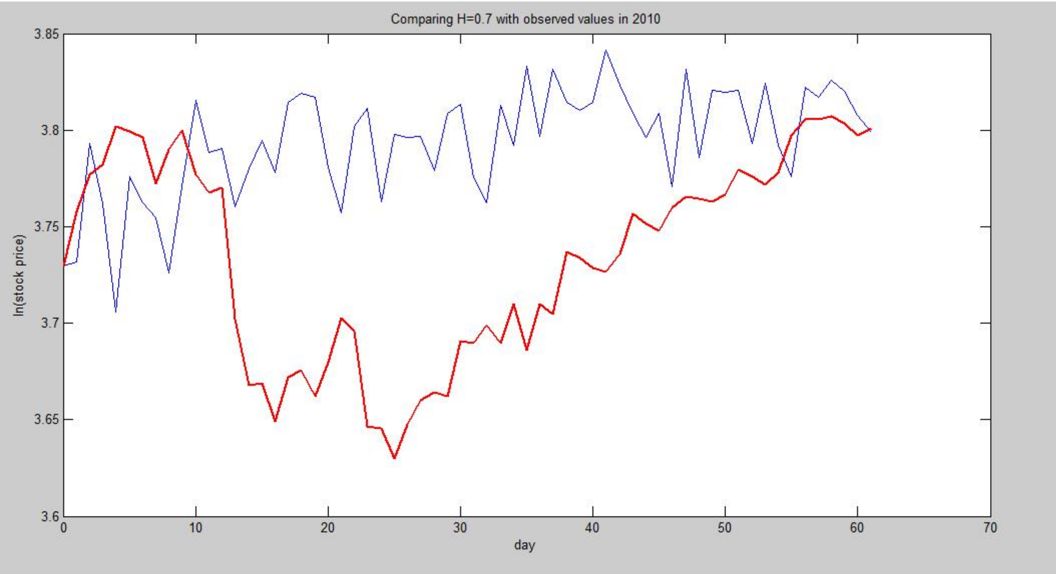

Comparing H=0.7 with observed values in 2009
ln(stock price)
day

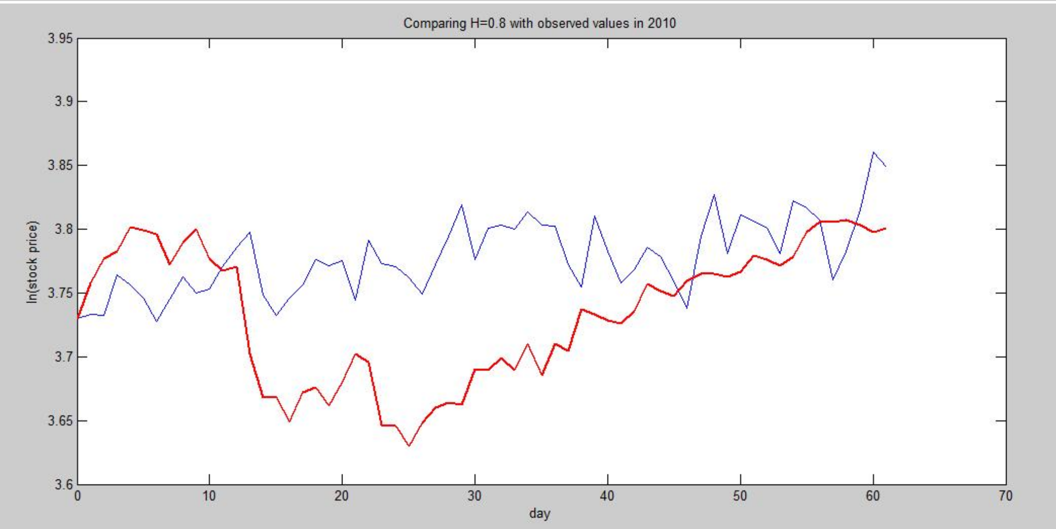

Comparing H=0.8 with observed values in 2009
ln(stock price)
day

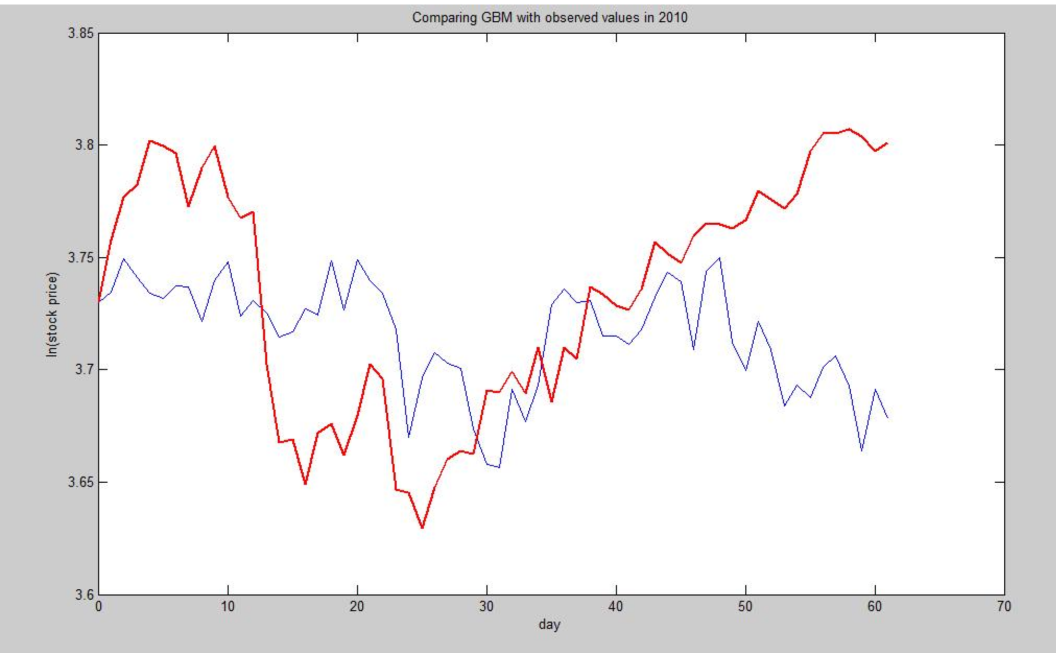

Comparing GBM with observed values in 2009
ln(stock price)
day

2010

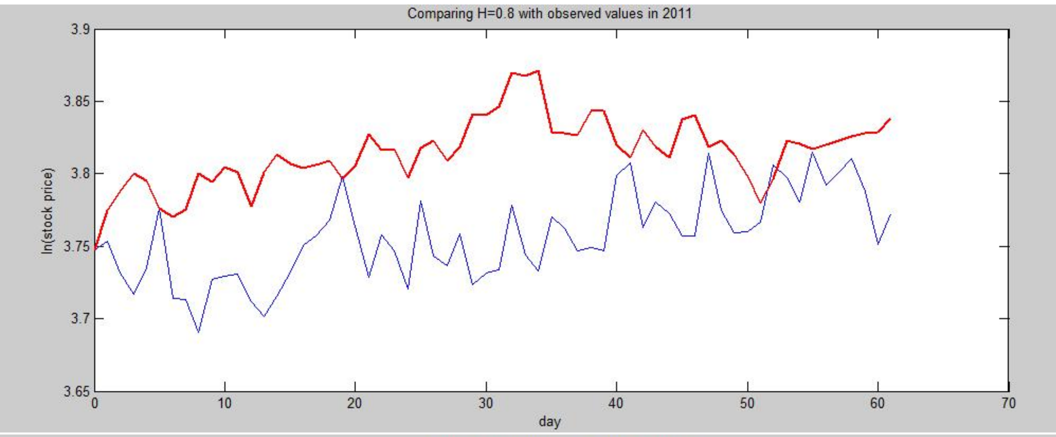

Comparing H=0.5 with observed values in 2010
ln(stock price)
3.6
3.65
3.7
3.75
3.8
3.85
0
10
20
30
40
50
60
70
day

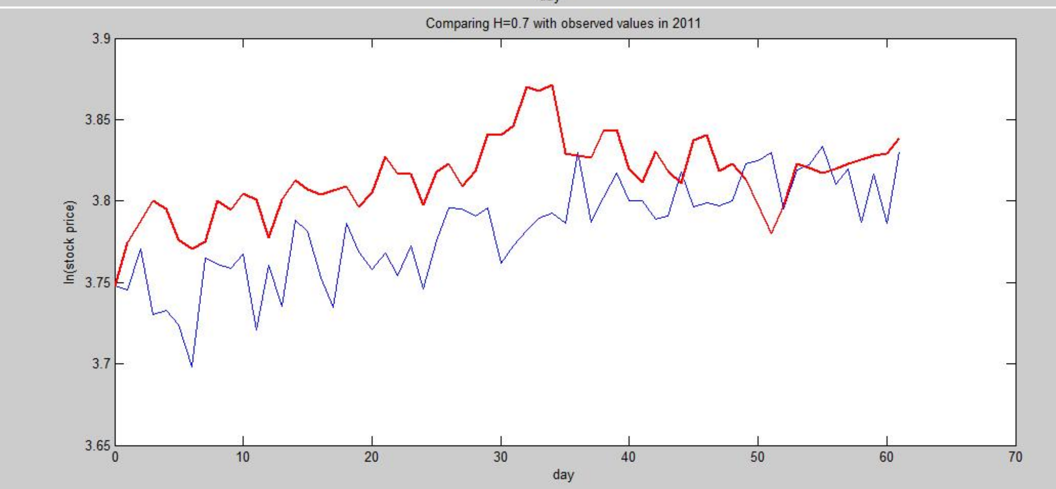

Comparing H=0.7 with observed values in 2010
ln(stock price)
3.6
3.65
3.7
3.75
3.8
3.85
0
10
20
30
40
50
60
70
day

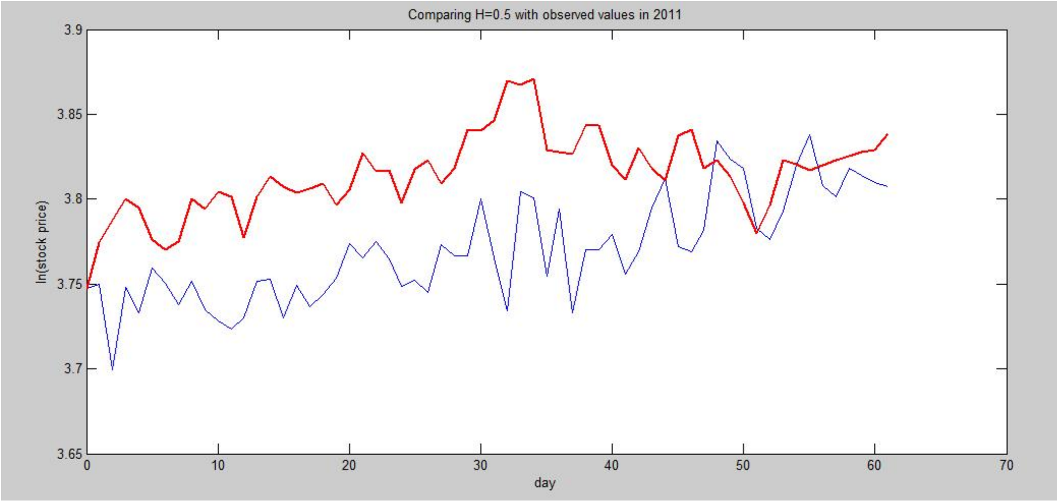

Comparing H=0.8 with observed values in 2010
ln(stock price)
3.6
3.65
3.7
3.75
3.8
3.85
3.9
3.95
0
10
20
30
40
50
60
70
day

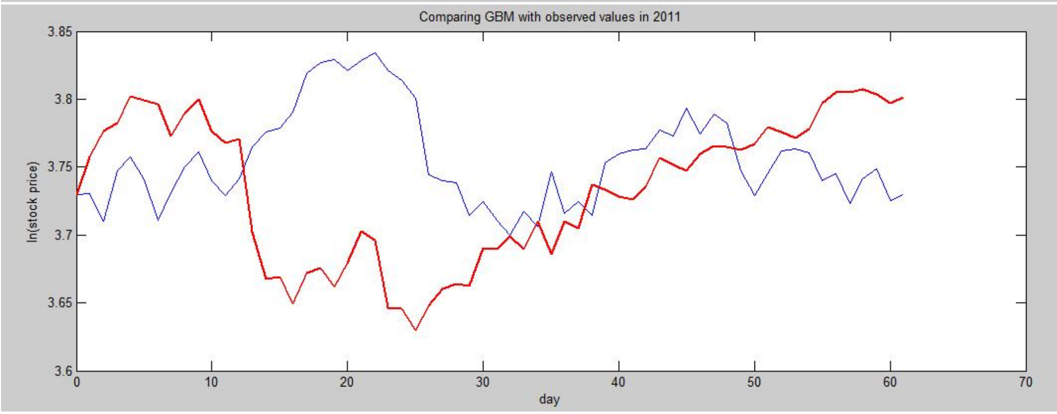

Comparing GBM with observed values in 2010
ln(stock price)
day

2011

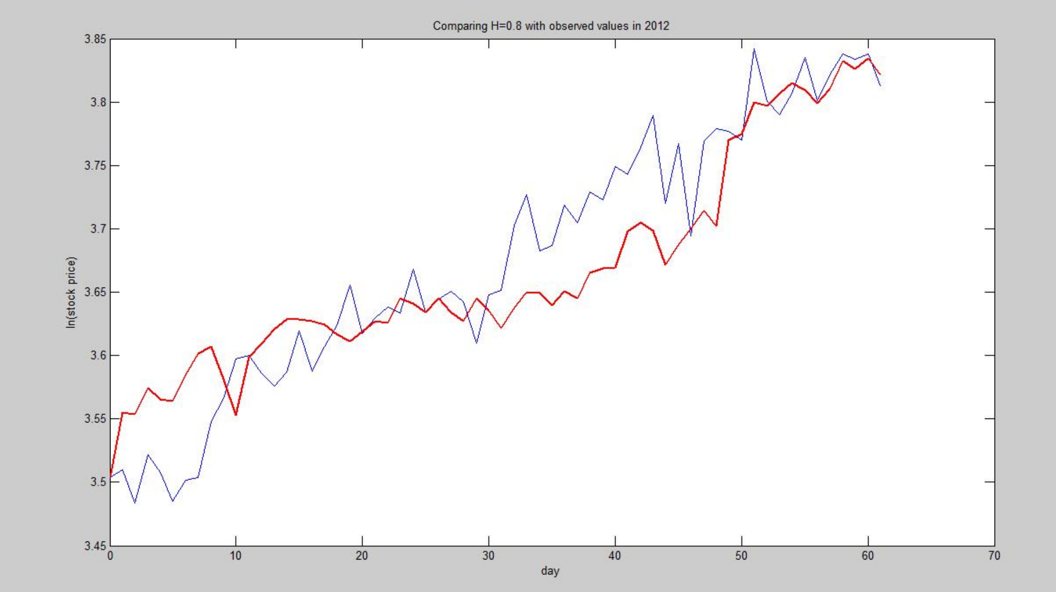

Comparing H=0.8 with observed values in 2011
ln(stock price)
day


Comparing H=0.7 with observed values in 2011
ln(stock price)
day

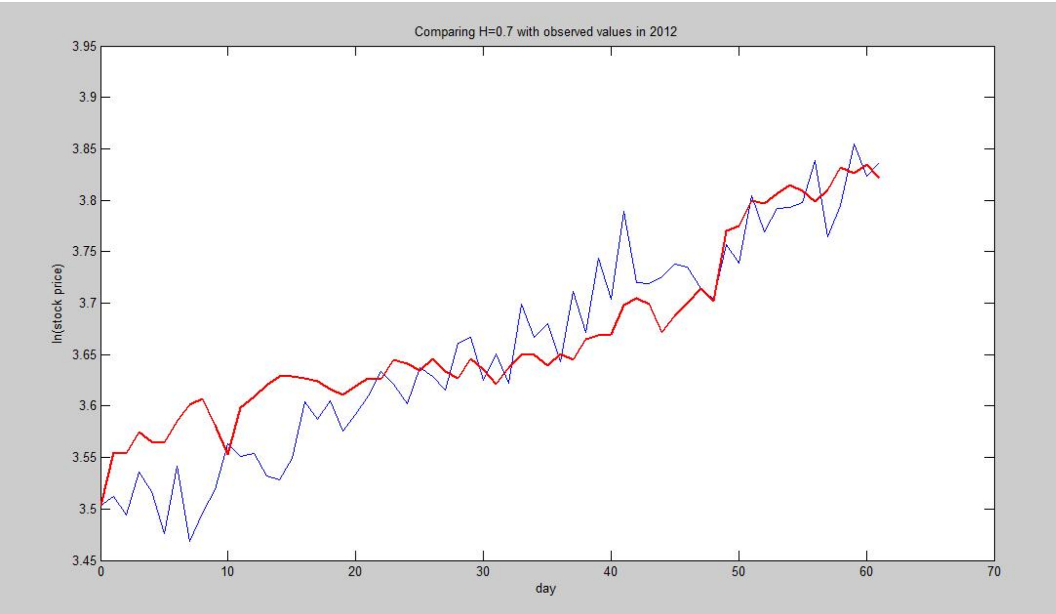

Comparing H=0.5 with observed values in 2011
ln(stock price)
day
3.9
3.85
3.8
3.75
3.7
3.65
0
10
20
30
40
50
60
70


Comparing GBM with observed values in 2011
ln(stock price)
day
3.85
3.8
3.75
3.7
3.65
3.6
0
10
20
30
40
50
60
70

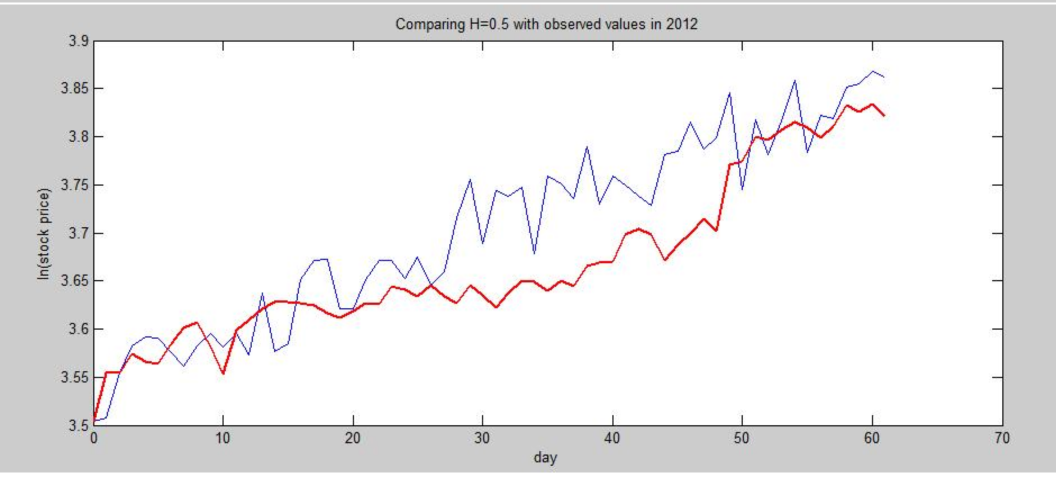
2012



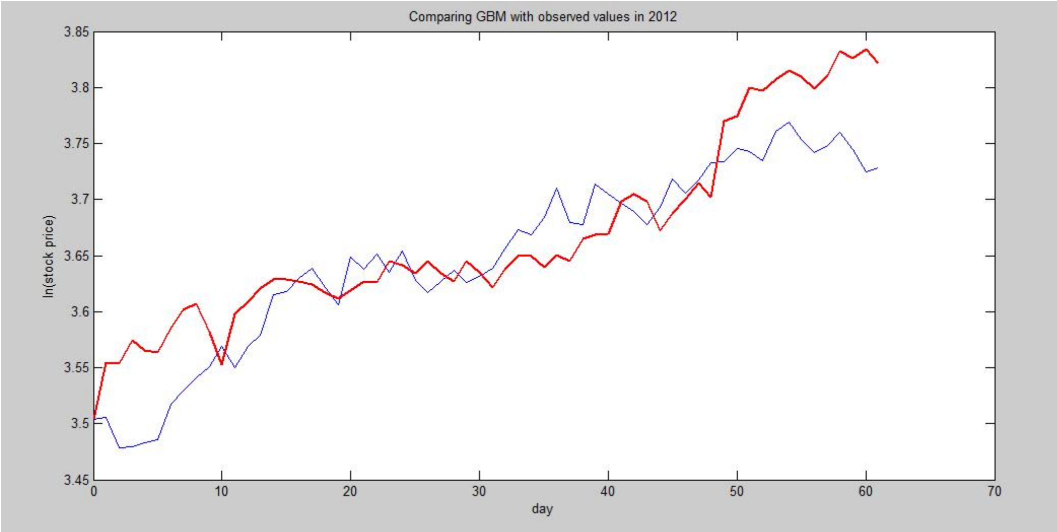
Comparing H=0.8 with observed values in 2012
ln(stock price)
day



Comparing H=0.7 with observed values in 2012
ln(stock price)
day



Comparing H=0.5 with observed values in 2012
ln(stock price)
day



9.2 Results from Comparing Errors between Stock Price Models

2007 Sample of 100 Norm Errors between Observed Price and Model

One-Sample Kolmogorov-Smirnov Test

		H=0.5 1norm	H=0.5 2 norm	H=0.5 inf norm	H=0.7 1norm	H=0.7 2norm	H=0.7 inf norm	H=0.8 1norm	H=0.8 2norm	H=0.8 inf norm	GBM1norm	GBM2norm	GBMinfnorm
N		100	100	100	100	100	100	100	100	100	100	100	100
Normal Parameters[a,b]	Mean	2.0018	.7658	.0914	2.0595	.3209	.0909	1.9567	.3053	.0888	3.2705	.4978	.1247
	Std. Deviation	.41249	.12886	.01499	.48436	.07073	.01731	.47564	.06930	.01618	1.91661	.26535	.04974
Most Extreme Differences	Absolute	.178	.179	.098	.186	.165	.064	.219	.203	.128	.186	.176	.094
	Positive	.178	.179	.098	.186	.165	.064	.219	.203	.128	.186	.176	.094
	Negative	-.125	-.085	-.061	-.122	-.113	-.043	-.151	-.124	-.041	-.122	-.112	-.071
Kolmogorov-Smirnov Z		1.780	1.790	.979	1.865	1.649	.637	2.190	2.026	1.276	1.863	1.760	.938
Asymp. Sig. (2-tailed)		.004	.003	.293	.002	.009	.812	.000	.001	.077	.002	.004	.342

a. Test distribution is Normal.

b. Calculated from data.

Paired Samples Test

		Paired Differences					t	df	Sig. (2-tailed)
		Mean	Std. Deviation	Std. Error Mean	95% Confidence Interval of the Difference Lower	Upper			
Pair 1	H=0.5 inf norm - GBMinfnorm	-.03324	.05314	.00531	-.04379	-.02270	-6.255	99	.000
Pair 2	H=0.7 inf norm - H=0.8 inf norm	.00207	.02522	.00252	-.00293	.00708	.821	99	.413
Pair 3	H=0.8 inf norm - GBMinfnorm	-.03587	.05261	.00526	-.04631	-.02543	-6.818	99	.000

2008 Sample of 100 Norm Errors between Observed Price and Model

One-Sample Kolmogorov-Smirnov Test

		H=0.5 1norm	H=0.5 2 norm	H=0.5 inf norm	H=0.7 1norm	H=0.7 2norm	H=0.7 inf norm	H=0.8 1norm	H=0.8 2norm	H=0.8 inf norm	GBM1norm	GBM2norm	GBMinfnorm
N		100	100	100	100	100	100	100	100	100	100	100	100
Normal Parameters[a,b]	Mean	4.8906	.9248	.2351	4.9811	.9586	.2397	4.9382	.7797	.2413	9.8147	1.4889	.3833
	Std. Deviation	.90739	.21183	.04002	1.16191	.23236	.04715	.96208	.13899	.04145	4.88538	.69674	.13746
Most Extreme Differences	Absolute	.191	.157	.096	.169	.176	.099	.134	.125	.080	.156	.152	.118
	Positive	.191	.157	.096	.169	.176	.099	.134	.125	.080	.156	.152	.118
	Negative	-.104	-.101	-.064	-.115	-.130	-.053	-.096	-.085	-.038	-.107	-.105	-.077
Kolmogorov-Smirnov Z		1.915	1.571	.956	1.691	1.756	.985	1.341	1.247	.797	1.562	1.524	1.184
Asymp. Sig. (2-tailed)		.001	.014	.320	.007	.004	.286	.055	.089	.550	.015	.019	.121

a. Test distribution is Normal.

b. Calculated from data.

Paired Samples Test

		Paired Differences					t	df	Sig. (2-tailed)
		Mean	Std. Deviation	Std. Error Mean	95% Confidence Interval of the Difference Lower	Upper			
Pair 1	H=0.5 inf norm - H=0.7 inf norm	-.00456	.05672	.00567	-.01581	.00670	-.804	99	.424
Pair 2	H=0.5 inf norm - GBMinfnorm	-.14823	.14356	.01436	-.17671	-.11974	-10.325	99	.000
Pair 3	H=0.7 inf norm - GBMinfnorm	-.14367	.14520	.01452	-.17248	-.11486	-9.894	99	.000

2009 Sample of 100 Norm Errors between Observed Price and Model

One-Sample Kolmogorov-Smirnov Test

		H=0.5 1norm	H=0.5 2 norm	H=0.5 inf norm	H=0.7 1norm	H=0.7 2norm	H=0.7 inf norm	H=0.8 1norm	H=0.8 2norm	H=0.8 inf norm	GBM1norm	GBM2norm	GBMinfnorm
N		100	100	100	100	100	100	100	100	100	100	100	100
Normal Parameters[a,b]	Mean	12.7939	1.9886	.6309	13.4360	2.0776	.6408	13.2936	2.0611	.6342	22.6189	3.4185	.9078
	Std. Deviation	3.76971	.48510	.11532	3.98504	.52075	.12823	3.85699	.50204	.11952	13.59191	1.82064	.34959
Most Extreme Differences	Absolute	.126	.121	.049	.142	.115	.066	.094	.089	.084	.177	.150	.123
	Positive	.126	.121	.049	.142	.115	.066	.094	.089	.084	.177	.150	.123
	Negative	-.124	-.106	-.035	-.097	-.081	-.044	-.071	-.059	-.062	-.147	-.132	-.096
Kolmogorov-Smirnov Z		1.258	1.207	.487	1.420	1.151	.660	.944	.888	.839	1.774	1.500	1.234
Asymp. Sig. (2-tailed)		.084	.108	.972	.036	.141	.776	.335	.409	.482	.004	.022	.095

a. Test distribution is Normal.

b. Calculated from data.

Paired Samples Test

		Paired Differences							
					95% Confidence Interval of the Difference				
		Mean	Std. Deviation	Std. Error Mean	Lower	Upper	t	df	Sig. (2-tailed)
Pair 1	H=0.5 inf norm - GBMinfnorm	-.27693	.36650	.03665	-.34965	-.20421	-7.556	99	.000
Pair 2	H=0.5 inf norm - H=0.8 inf norm	-.00329	.15124	.01512	-.03330	.02672	-.217	99	.828
Pair 3	H=0.8 inf norm - GBMinfnorm	-.27364	.37091	.03709	-.34724	-.20005	-7.378	99	.000

2010 Sample of 100 Norm Errors between Observed Price and Model

One-Sample Kolmogorov-Smirnov Test

		H=0.5 1norm	H=0.5 2 norm	H=0.5 inf norm	H=0.7 1norm	H=0.7 2norm	H=0.7 inf norm	H=0.8 1norm	H=0.8 2norm	H=0.8 inf norm	GBM1norm	GBM2norm	GBMinfnorm
N		100	100	100	100	100	100	100	100	100	100	100	100
Normal Parameters[a,b]	Mean	3.4548	.5324	.1813	3.5719	.5479	.1540	3.5362	.5421	.1522	3.8020	.5902	.1749
	Std. Deviation	.69039	.10023	.03541	.68412	.10055	.02626	.63334	.09247	.02511	.81001	.11723	.03174
Most Extreme Differences	Absolute	.147	.165	.105	.122	.122	.127	.145	.147	.075	.134	.137	.095
	Positive	.147	.165	.105	.122	.122	.127	.145	.147	.075	.121	.137	.095
	Negative	-.106	-.113	-.046	-.104	-.099	-.058	-.100	-.093	-.057	-.134	-.115	-.060
Kolmogorov-Smirnov Z		1.467	1.646	1.046	1.222	1.221	1.269	1.455	1.470	.751	1.343	1.366	.952
Asymp. Sig. (2-tailed)		.027	.009	.224	.101	.102	.080	.029	.027	.626	.054	.048	.325

a. Test distribution is Normal.

b. Calculated from data.

Paired Samples Test

		Paired Differences							
					95% Confidence Interval of the Difference				
		Mean	Std. Deviation	Std. Error Mean	Lower	Upper	t	df	Sig. (2-tailed)
Pair 1	H=0.5 inf norm - GBMinfnorm	.00636	.04487	.00449	-.00255	.01526	1.416	99	.160
Pair 2	H=0.8 inf norm - H=0.7 inf norm	-.00185	.03684	.00368	-.00916	.00546	-.501	99	.617
Pair 3	H=0.8 inf norm - GBMinfnorm	-.02274	.03945	.00394	-.03057	-.01492	-5.766	99	.000

2011 Sample of 100 Norm Errors between Observed Price and Model

One-Sample Kolmogorov-Smirnov Test

		H=0.5 1norm	H=0.5 2 norm	H=0.5 inf norm	H=0.7 1norm	H=0.7 2norm	H=0.7 infnorm	H=0.8 1norm	H=0.8 2norm	H=0.8 infnorm	GBM1norm	GBM2norm	GBMinfnorm
N		100	100	100	100	100	100	100	100	100	100	100	100
Normal Parameters[a,b]	Mean	2.2604	.3436	.1032	2.2340	.3384	.1009	2.1892	.3346	.0997	3.9846	.5884	.1462
	Std. Deviation	.71219	.09142	.02104	.80312	.10371	.02207	.69608	.09250	.02253	2.07684	.28122	.05391
Most Extreme Differences	Absolute	.133	.126	.067	.180	.184	.086	.151	.148	.099	.165	.169	.116
	Positive	.133	.126	.067	.180	.184	.086	.151	.148	.099	.165	.169	.116
	Negative	-.097	-.087	-.042	-.133	-.126	-.054	-.096	-.086	-.077	-.122	-.108	-.080
Kolmogorov-Smirnov Z		1.327	1.256	.667	1.798	1.841	.857	1.514	1.479	.991	1.654	1.693	1.162
Asymp. Sig. (2-tailed)		.059	.085	.766	.003	.002	.455	.020	.025	.280	.008	.006	.134

a. Test distribution is Normal.

b. Calculated from data.

Paired Samples Test

		Paired Differences							
					95% Confidence Interval of the Difference				
		Mean	Std. Deviation	Std. Error Mean	Lower	Upper	t	df	Sig. (2-tailed)
Pair 1	H=0.8 infnorm - H=0.7 infnorm	-.00118	.03004	.00300	-.00714	.00478	-.394	99	.695
Pair 2	H=0.8 infnorm - GBMinfnorm	-.04649	.05593	.00559	-.05759	-.03539	-8.311	99	.000
Pair 3	H=0.5 inf norm - GBMinfnorm	-.04295	.05606	.00561	-.05408	-.03183	-7.662	99	.000

2012 Sample of 100 Norm Errors between Observed Price and Model

One-Sample Kolmogorov-Smirnov Test

		H=0.5 1norm	H=0.5 2 norm	H=0.5 inf norm	H=0.7 1norm	H=0.7 2 norm	H=0.7 infnorm	H=0.8 1norm	H=0.8 2norm	H=0.8 inf norm	GBM1norm	GBM2norm	GBMinfnorm
N		100	100	100	100	100	100	100	100	100	100	100	100
Normal Parameters[a,b]	Mean	2.4842	.3869	.1144	2.5111	.3922	.1175	2.5648	.3996	.1176	4.8361	.7186	.1736
	Std. Deviation	.37965	.05803	.01989	.42651	.06260	.01798	.61405	.08550	.02179	2.68546	.37147	.06991
Most Extreme Differences	Absolute	.150	.142	.091	.151	.136	.099	.170	.164	.130	.150	.151	.117
	Positive	.150	.142	.091	.151	.136	.099	.170	.164	.130	.150	.151	.117
	Negative	-.075	-.085	-.047	-.090	-.081	-.040	-.137	-.124	-.067	-.117	-.116	-.079
Kolmogorov-Smirnov Z		1.502	1.424	.913	1.510	1.356	.987	1.701	1.641	1.296	1.500	1.513	1.172
Asymp. Sig. (2-tailed)		.022	.035	.375	.021	.050	.284	.006	.009	.070	.022	.020	.128

a. Test distribution is Normal.

b. Calculated from data.

Paired Samples Test

		Paired Differences							
					95% Confidence Interval of the Difference				
		Mean	Std. Deviation	Std. Error Mean	Lower	Upper	t	df	Sig. (2-tailed)
Pair 1	H=0.5 inf norm - H=0.7 infnorm	-.00308	.02726	.00273	-.00849	.00233	-1.131	99	.261
Pair 2	H=0.5 inf norm - GBMinfnorm	-.05912	.07465	.00746	-.07394	-.04431	-7.920	99	.000
Pair 3	H=0.8 inf norm - H=0.5 inf norm	.00319	.03015	.00302	-.00279	.00918	1.060	99	.292

9.3 Results from Predicting Future Values from Short Term Models (10 days)

2007 Short Term Prediction

One-Sample Kolmogorov-Smirnov Test

		H=0.5	H=0.7	H=0.8	GBM1
N		100	100	100	100
Normal Parameters[a,b]	Mean	.0231	.0212	.0209	.0170
	Std. Deviation	.01519	.01565	.01799	.01363
Most Extreme Differences	Absolute	.086	.130	.125	.108
	Positive	.086	.130	.116	.103
	Negative	-.067	-.094	-.125	-.108
Kolmogorov-Smirnov Z		.857	1.300	1.251	1.076
Asymp. Sig. (2-tailed)		.454	.068	.088	.197

a. Test distribution is Normal.

b. Calculated from data.

Paired Samples Test

		Paired Differences					t	df	Sig. (2-tailed)
					95% Confidence Interval of the Difference				
		Mean	Std. Deviation	Std. Error Mean	Lower	Upper			
Pair 1	GBM1 - H=0.5	-.00609	.02033	.00203	-.01013	-.00206	-2.996	99	.003
Pair 2	H=0.7 - H=0.5	-.00184	.02267	.00227	-.00634	.00266	-.812	99	.418
Pair 3	H=0.7 - H=0.8	.00035	.02322	.00232	-.00426	.00496	.150	99	.881

2008 Short Term Prediction

One-Sample Kolmogorov-Smirnov Test

		H=0.5	H=0.7	H=0.8	GBM1
N		100	100	100	100
Normal Parameters[a,b]	Mean	.1290	.1291	.1122	.0938
	Std. Deviation	.09329	.08904	.10128	.06710
Most Extreme Differences	Absolute	.116	.101	.156	.086
	Positive	.116	.101	.156	.084
	Negative	-.084	-.077	-.134	-.086
Kolmogorov-Smirnov Z		1.163	1.006	1.557	.864
Asymp. Sig. (2-tailed)		.134	.263	.016	.444

a. Test distribution is Normal.

b. Calculated from data.

Paired Samples Test

		Paired Differences					t	df	Sig. (2-tailed)
					95% Confidence Interval of the Difference				
		Mean	Std. Deviation	Std. Error Mean	Lower	Upper			
Pair 1	H=0.8 - GBM1	.01845	.12731	.01273	-.00681	.04371	1.450	99	.150
Pair 2	H=0.8 - H=0.7	-.01692	.14307	.01431	-.04530	.01147	-1.182	99	.240
Pair 3	H=0.8 - H=0.5	-.01674	.13652	.01365	-.04382	.01035	-1.226	99	.223

2009 Short Term Prediction

One-Sample Kolmogorov-Smirnov Test

		H=0.5	H=0.7	H=0.8	GBM1
N		100	100	100	100
Normal Parameters[a,b]	Mean	.1647	.1579	.1526	.1022
	Std. Deviation	.11349	.10856	.10551	.07899
Most Extreme Differences	Absolute	.095	.104	.104	.102
	Positive	.095	.104	.104	.094
	Negative	-.080	-.073	-.074	-.102
Kolmogorov-Smirnov Z		.951	1.035	1.042	1.017
Asymp. Sig. (2-tailed)		.327	.234	.228	.252

a. Test distribution is Normal.

b. Calculated from data.

Paired Samples Test

		Paired Differences							
					95% Confidence Interval of the Difference				
		Mean	Std. Deviation	Std. Error Mean	Lower	Upper	t	df	Sig. (2-tailed)
Pair 1	H=0.8 - H=0.7	-.00531	.15294	.01529	-.03566	.02503	-.347	99	.729
Pair 2	H=0.8 - GBM1	.05048	.12373	.01237	.02593	.07503	4.080	99	.000
Pair 3	H=0.5 - GBM1	.06259	.13766	.01377	.03527	.08990	4.546	99	.000
Pair 4	H=0.8 - H=0.5	-.01211	.14451	.01445	-.04078	.01657	-.838	99	.404

2010 Short Term Prediction

One-Sample Kolmogorov-Smirnov Test

		H=0.5	H=0.7	H=0.8	GBM1
N		100	100	100	100
Normal Parameters[a,b]	Mean	.0659	.0687	.0733	.0539
	Std. Deviation	.04940	.05731	.04792	.04256
Most Extreme Differences	Absolute	.116	.120	.094	.105
	Positive	.116	.120	.094	.104
	Negative	-.094	-.120	-.066	-.105
Kolmogorov-Smirnov Z		1.159	1.202	.943	1.046
Asymp. Sig. (2-tailed)		.136	.111	.336	.224

a. Test distribution is Normal.

b. Calculated from data.

Paired Samples Test

		Paired Differences							
					95% Confidence Interval of the Difference				
		Mean	Std. Deviation	Std. Error Mean	Lower	Upper	t	df	Sig. (2-tailed)
Pair 1	H=0.5 - H=0.7	-.00276	.07510	.00751	-.01767	.01214	-.368	99	.714
Pair 2	H=0.5 - H=0.8	-.00743	.07094	.00709	-.02151	.00664	-1.048	99	.297
Pair 3	H=0.5 - GBM1	.01206	.06443	.00644	-.00073	.02484	1.871	99	.064

2011 Short Term Prediction

One-Sample Kolmogorov-Smirnov Test

		H=0.5	H=0.7	H=0.8	GBM1
N		100	100	100	100
Normal Parameters[a,b]	Mean	.0515	.0501	.0544	.0468
	Std. Deviation	.03581	.04054	.04083	.03405
Most Extreme Differences	Absolute	.077	.109	.121	.094
	Positive	.077	.107	.121	.094
	Negative	-.075	-.109	-.092	-.088
Kolmogorov-Smirnov Z		.771	1.085	1.212	.941
Asymp. Sig. (2-tailed)		.592	.190	.106	.339

a. Test distribution is Normal.

b. Calculated from data.

Paired Samples Test

		Paired Differences							
					95% Confidence Interval of the Difference				
		Mean	Std. Deviation	Std. Error Mean	Lower	Upper	t	df	Sig. (2-tailed)
Pair 1	H=0.7 - H=0.5	-.00140	.05540	.00554	-.01239	.00959	-.253	99	.801
Pair 2	H=0.7 - H=0.8	-.00423	.05758	.00576	-.01565	.00720	-.734	99	.465
Pair 3	H=0.7 - GBM1	.00334	.05513	.00551	-.00760	.01428	.606	99	.546
Pair 4	H=0.5 - GBM1	.00474	.04529	.00453	-.00425	.01373	1.047	99	.298

2012 Short Term Prediction

One-Sample Kolmogorov-Smirnov Test

		H=0.5	H=0.7	H=0.8	GBM1
N		100	100	100	100
Normal Parameters[a,b]	Mean	.1021	.0876	.0993	.0678
	Std. Deviation	.07795	.06477	.07291	.04812
Most Extreme Differences	Absolute	.111	.108	.095	.114
	Positive	.111	.108	.095	.114
	Negative	-.099	-.090	-.088	-.080
Kolmogorov-Smirnov Z		1.113	1.078	.954	1.139
Asymp. Sig. (2-tailed)		.168	.195	.323	.149

a. Test distribution is Normal.

b. Calculated from data.

Paired Samples Test

		Paired Differences					t	df	Sig. (2-tailed)
					95% Confidence Interval of the Difference				
		Mean	Std. Deviation	Std. Error Mean	Lower	Upper			
Pair 1	H=0.7 - H=0.5	-.01449	.09451	.00945	-.03324	.00426	-1.533	99	.128
Pair 2	H=0.7 - H=0.8	-.01163	.09674	.00967	-.03082	.00757	-1.202	99	.232
Pair 3	H=0.7 - GBM1	.01987	.08142	.00814	.00372	.03603	2.441	99	.016
Pair 4	H=0.5 - GBM1	.03436	.08855	.00885	.01679	.05193	3.881	99	.000

9.4 Results from Predicting Future Values from Long Term Models (60 days)

2007 Long Term Prediction

One-Sample Kolmogorov-Smirnov Test

		H=0.5	H=0.7	H=0.8	GBM1
N		100	100	100	100
Normal Parameters[a,b]	Mean	.0082	.0091	.0091	.0717
	Std. Deviation	.00589	.00705	.00667	.05326
Most Extreme Differences	Absolute	.109	.104	.101	.092
	Positive	.109	.100	.101	.092
	Negative	-.081	-.104	-.091	-.090
Kolmogorov-Smirnov Z		1.086	1.037	1.010	.920
Asymp. Sig. (2-tailed)		.189	.233	.259	.366

a. Test distribution is Normal.

b. Calculated from data.

Paired Samples Test

		Paired Differences					t	df	Sig. (2-tailed)
					95% Confidence Interval of the Difference				
		Mean	Std. Deviation	Std. Error Mean	Lower	Upper			
Pair 1	GBM1 - H=0.5	.06343	.05367	.00537	.05278	.07407	11.817	99	.000
Pair 2	H=0.5 - H=0.7	-.00088	.00879	.00088	-.00262	.00086	-1.002	99	.319
Pair 3	H=0.5 - H=0.8	-.00088	.00841	.00084	-.00255	.00079	-1.045	99	.299

2008 Long Term Prediction

One-Sample Kolmogorov-Smirnov Test

		H=0.5	H=0.7	H=0.8	GBM1
N		100	100	100	100
Normal Parameters[a,b]	Mean	.0594	.0674	.0559	.2149
	Std. Deviation	.04166	.04353	.04671	.18220
Most Extreme Differences	Absolute	.086	.092	.134	.121
	Positive	.086	.092	.134	.099
	Negative	-.082	-.068	-.116	-.121
Kolmogorov-Smirnov Z		.857	.921	1.341	1.206
Asymp. Sig. (2-tailed)		.454	.364	.055	.109

a. Test distribution is Normal.

b. Calculated from data.

Paired Samples Test

		Paired Differences					t	df	Sig. (2-tailed)
		Mean	Std. Deviation	Std. Error Mean	95% Confidence Interval of the Difference Lower	Upper			
Pair 1	H=0.8 - H=0.5	-.00347	.06086	.00609	-.01554	.00861	-.570	99	.570
Pair 2	H=0.8 - H=0.7	-.01152	.06069	.00607	-.02356	.00053	-1.897	99	.061
Pair 3	H=0.8 - GBM1	-.15904	.19066	.01907	-.19687	-.12121	-8.341	99	.000
Pair 4	H=0.5 - GBM1	-.15557	.19275	.01928	-.19382	-.11732	-8.071	99	.000

2009 Long Term Prediction

One-Sample Kolmogorov-Smirnov Test

		H=0.5	H=0.7	H=0.8	GBM1
N		100	100	100	100
Normal Parameters[a,b]	Mean	.1368	.1535	.1543	.4769
	Std. Deviation	.10182	.10665	.12738	.34282
Most Extreme Differences	Absolute	.123	.128	.127	.099
	Positive	.123	.128	.127	.099
	Negative	-.090	-.084	-.122	-.083
Kolmogorov-Smirnov Z		1.227	1.285	1.270	.991
Asymp. Sig. (2-tailed)		.098	.074	.079	.279

a. Test distribution is Normal.

b. Calculated from data.

Paired Samples Test

		Paired Differences					t	df	Sig. (2-tailed)
		Mean	Std. Deviation	Std. Error Mean	95% Confidence Interval of the Difference Lower	Upper			
Pair 1	H=0.5 - H=0.7	-.01672	.14481	.01448	-.04545	.01202	-1.154	99	.251
Pair 2	H=0.5 - H=0.8	-.01754	.15918	.01592	-.04913	.01404	-1.102	99	.273
Pair 3	H=0.5 - GBM1	-.34010	.36247	.03625	-.41202	-.26818	-9.383	99	.000

2010 Long Term Prediction

One-Sample Kolmogorov-Smirnov Test

		H=0.5	H=0.7	H=0.8	GBM1
N		100	100	100	100
Normal Parameters[a,b]	Mean	.0289	.0269	.0282	.2154
	Std. Deviation	.02067	.01818	.02326	.15619
Most Extreme Differences	Absolute	.121	.098	.123	.104
	Positive	.121	.098	.118	.104
	Negative	-.082	-.077	-.123	-.088
Kolmogorov-Smirnov Z		1.210	.985	1.229	1.036
Asymp. Sig. (2-tailed)		.107	.287	.098	.233

a. Test distribution is Normal.

b. Calculated from data.

Paired Samples Test

		Paired Differences					t	df	Sig. (2-tailed)
					95% Confidence Interval of the Difference				
		Mean	Std. Deviation	Std. Error Mean	Lower	Upper			
Pair 1	GBM1 - H=0.5	.06343	.05367	.00537	.05278	.07407	11.817	99	.000
Pair 2	H=0.5 - H=0.7	-.00088	.00879	.00088	-.00262	.00086	-1.002	99	.319
Pair 3	H=0.7 - H=0.8	.00000	.00885	.00089	-.00176	.00176	.001	99	.999
Pair 4	H=0.7 - GBM1	-.06255	.05388	.00539	-.07324	-.05185	-11.609	99	.000

2011 Long Term Prediction

One-Sample Kolmogorov-Smirnov Test

		H=0.5	H=0.7	H=0.8	GBM1
N		100	100	100	100
Normal Parameters[a,b]	Mean	.0225	.0261	.0281	.1013
	Std. Deviation	.01837	.01843	.01978	.07202
Most Extreme Differences	Absolute	.133	.094	.115	.100
	Positive	.133	.094	.115	.100
	Negative	-.116	-.091	-.087	-.080
Kolmogorov-Smirnov Z		1.330	.938	1.153	1.002
Asymp. Sig. (2-tailed)		.058	.342	.140	.268

a. Test distribution is Normal.

b. Calculated from data.

Paired Samples Test

		Paired Differences					t	df	Sig. (2-tailed)
					95% Confidence Interval of the Difference				
		Mean	Std. Deviation	Std. Error Mean	Lower	Upper			
Pair 1	H=0.5 - H=0.7	-.00362	.02963	.00296	-.00950	.00226	-1.223	99	.224
Pair 2	H=0.5 - H=0.8	-.00568	.02927	.00293	-.01149	.00013	-1.940	99	.055
Pair 3	H=0.5 - GBM1	-.07886	.07639	.00764	-.09402	-.06370	-10.323	99	.000

2012 Long Term Prediction

One-Sample Kolmogorov-Smirnov Test

		H=0.5	H=0.7	H=0.8	GBM1
N		100	100	100	100
Normal Parameters[a,b]	Mean	.0331	.0282	.0606	.1094
	Std. Deviation	.02557	.02192	.04760	.07857
Most Extreme Differences	Absolute	.113	.102	.105	.132
	Positive	.113	.097	.105	.132
	Negative	-.113	-.102	-.101	-.086
Kolmogorov-Smirnov Z		1.133	1.024	1.051	1.317
Asymp. Sig. (2-tailed)		.153	.245	.219	.062

a. Test distribution is Normal.

b. Calculated from data.

Paired Samples Test

		Paired Differences					t	df	Sig. (2-tailed)
					95% Confidence Interval of the Difference				
		Mean	Std. Deviation	Std. Error Mean	Lower	Upper			
Pair 1	H=0.7 - H=0.5	-.00493	.03144	.00314	-.01117	.00131	-1.568	99	.120
Pair 2	H=0.7 - H=0.8	-.03242	.05266	.00527	-.04287	-.02197	-6.156	99	.000
Pair 3	H=0.7 - GBM1	-.08124	.08194	.00819	-.09750	-.06499	-9.915	99	.000
Pair 4	H=0.5 - GBM1	-.07632	.08207	.00821	-.09260	-.06003	-9.298	99	.000

9.5 MATLAB Code for Optimising Option Pricing

```
stock=xlsread('stock.xlsx') ;
strike=xlsread('strike.xlsx');
time=xlsread('time.xlsx');
midpoint=xlsread('midpoint.xlsx');
N=170;
d1=zeros(N,1);
d2=zeros(N,1);
callprice=zeros(N,1);
absoluteerror=zeros(N,1);
steps=60;
averageerror=zeros(steps,steps);
t=(0.6-0.3)/steps;
for index=1:steps+1
H=0.3+(index-1)*t;
for k=1:steps+1
sigma=0.10+(k-1)*t;
for g=1:N
d1(g,1)=(log(stock(g,1)/strike(g,1))+0.05*time(g,1))/((time(g,1)^(H))*sigma)+1/2*sigma*(time(g,1)^H);
d2(g,1)=d1(g,1)-sigma*time(g,1).^H;
callprice(g,1) = stock(g,1)*normcdf(d1(g,1))-strike(g,1)*exp(-0.05*time(g,1))*normcdf(d2(g,1));
absoluteerror(g,1)= abs(callprice(g,1)- midpoint(g,1));
end
averageerror(index,k)=mean(absoluteerror);
end
end

H=meshgrid(0.3:t:0.6);
sigma=meshgrid(0.10:t:0.10+(steps)*t);
mesh(H,sigma,averageerror)
xlabel('H')
ylabel('sigma')
zlabel('average error')
hold on
plot3(0.47,0.215,0.2905,'*')
```

```
for k=1:61
    smallestrowelement(k,1)=min(averageerror(k,:))
end
   [val,ind]=min(smallestrowelement)
   row=ind
 A=averageerror(row,:);
 [val,ind]=min(A);
 column=ind;
 H=0.3+(row-1)*t
 sigma=0.10+(column-1)*t
```

9.6 Notation

Notations:

Ω ... Wienerspace $C[0,1]$ resp. $C([0,1],\mathbb{R}^m)$
$\mathcal{F}$ ... natural filtration
H ... $L^2[0,1]$ resp. $L^2([0,1],\mathbb{R}^m)$
$H^{\otimes k}$ ... tensorproduct $\cong L^2([0,1]^k)$, $H^{\hat{\otimes} k}$ ... symmetric tensorproduct
$\tilde{H}$ ... Cameron-Martin-space $\subset \Omega$, elements are paths with derivative in H
$W : \mathcal{F} \to \mathbb{R}$ ... Wiener-measure on Ω
$\beta_t = \beta(t)$ ... Brownian Motion (= coordinate process on $(\Omega, \mathcal{F}, W)$)
$W : H \to L^2(\Omega)$ ... defined by $W(h) = \int_0^1 h d\beta$
$\mathcal{S}_2$ ... Wiener polynomials, functionals of form polynomial($W(h_1), ..., W(H_n)$
$\mathcal{S}_1$ ... cylindrical functionals, $\subset \mathcal{S}_2$
$\mathcal{D}^{k,p}$ ... $\subset L^p(\Omega)$ containing k-times Malliavin differentiable functionals
$\mathcal{D}^\infty$ ... $\cap_{k,p}\mathcal{D}^{k,p}$, smooth Wiener functionals
λ, λ^m ... (m-dimensional) Lebesgue-measure
ν, ν^n ... (n-dimensional) standard Gaussian measure
∇ ... gradient-operator on $\mathbb{R}^n$
$L^p(\Omega, H)$ ... H-valued random-variables s.t. $\int_\Omega \|\cdot\|_H dW < \infty$
D ... Malliavin derivative, operator $L^p(\Omega) \to L^p(\Omega, H)$
δ ... $= D^*$ the adjoint operator, also: divergence, Skorohod Integral
L ... $= \delta \circ D$, Ornstein-Uhlenbeck operator $L^p(\Omega) \to L^p(\Omega)$
$W^{k,p}$ ... Sobolev-spaces built on $\mathbb{R}^n$ H^k...$W^{k,2}$
∂ ... (for functions $f : \mathbb{R} \to \mathbb{R}$) simple differentiation
∂^* ... adjoint of ∂ on $L^2(\mathbb{R}, \nu)$
$\mathcal{L}$ ... $= \partial^* \partial$, one-dimensional OU-operator
$\partial_i, \partial_{ij}$ ... partial derivatices w.r.t. x_i, x_j etc
$\mathcal{L}$ ... generator of m-dimensional diffusion process, for instance $\mathcal{L} = E^{ij}\partial_{ij} + B^i\partial_i$
H_n ... Hermite-polynomials
$\Delta_n(t)$ ... n-dimensional simplex $\{0 < t_1 < ... < t_n < t\} \subset [0,1]^n$
$J(\cdot)$ ... Iterated Wiener-Ito integral, operator $L^2[\Delta_n]$ $to\mathcal{C}_n \subset L^2(\Omega)$
$\mathcal{C}_n$ ... n^{th} Wiener Chaos
α ... multiinex (finite-dimensional)
X ... m-dimensional diffusion process given by SDE, driven by d BMs
$\Lambda = \Lambda(X)$ ... $< DX, DX >_H$, Malliavin covariance matrix
V, W ... vectorfields on $\mathbb{R}^m$, seen as map $\mathbb{R}^m \to \mathbb{R}^m$ or as first order differential operator

References

[1] E. Ales, O. Mazet and D. Nualart (2001): Stochastic calculus with respect to Gaussian processes. Annals of Probability 29, 766-801.

[2] C. A. Ball and A. Roma (1994): Stochastic volatility option pricing.Journal of Financial and Quantitative Analysis 29, 589-607.

[3] Jeremy j. Becnel and ambar n. Sengupta. The schwartz space: a background to white noise analysis , 1st September 2004

[4] Biagini, Oksendal, Bernt (2008). Forward integrals and an Ito formula for fractional Brownian motion. Infinite Dimensional Analysis Quantum Probability and Related Topics. ISSN 0219-0257. 11, s 157- 177

[5] J Beran, Statistics for long-memory processes, no. 61 in Monographs on statistics and applied probability,Chapman and Hall, 1994.

[6] J. Beran, A goodness-of-fit test for time series with long range dependence, J. R. Statist. Soc. B, 54 (1992), pp. 749760

[7] Corrado, C. and T. Su (1996) Skewness and Kurtosis in S and P 500 Index Returns Implied by Option Prices, Journal of Financial Research, 19, 175192.

[8] P. Carmona, L. Coutin and G. Montseny (2003): Stochastic integration with respect to the fractional Brownian motion. Ann. Institut Henri Poincare 39 (1), 27-68.

[9] P. Cheridito (2001): Regularizing fractional Brownian motion with a view towards stock price modelling. PhD Dissertation.

[10] Ton Dieker, Simulation of fractional Brownian motion, Masters Thesis, Department of Mathematical Sciences, University of Twente, The Netherlands, 2004.

[11] Tyrone E. Duncan, Yaozhong Hu and Bozenna Pasik-Duncan, Stochastic Calculus for Fractional Brownian Motion Integration Theory19 Dec 2010, SIAMCO

[12] Samuel Dupernex, Student Economic Review, Vol. 21, 2007, Introduction Random walks and the Efficient Market Hypothesis-why might shares follow a random walk?

[13] Peter K . Friz, An Introduction to Malliavin Calculus Courant Institute of Mathematical Sciences New York University ,August 3, 2002 Department of Mathematics and Statistics, McMaster University,

[14] Robert M. Gray, Toeplitz and Circulant Matrices: A review, Deptartment of Electrical Engineering, Stanford University, Stanford 94305, USA, last modified 19 August 2013.

[15] Desmond J. Higham, An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations m, 2001 Society for Industrial and Applied Mathematics.

[16] Ross Ihaka,Time Series Analysis,Statistics Department, University of Auckland, April 14, 2005

[17] Joachim Lebovits Jacques Levy-Vehel, White Noise-based Stochastic Calculus with respect to multifractional Brownian motion , March 28, 2011

[18] D. Nualart: A White noise approach to fractional Brownian motion, in Stochastic Analysis: Classical and Quantum, Perspectives of White Noise Theory", T. Hida Ed., World Scientist 2005, 112-126

[19] D. Nualart, Finance and Stochastics , 533-553.

[20] D. Nualart, Fractional Brownian Motion,stochastic calculus and applications. (2003): Stochastic calculus with respect to the fractional Brownian motion and applications. Contemporary Mathematics 336, 3-39.

[21] B. Oksendal (2004): Fractional Brownian motion in Finance. Preprint. E. Renault and N. Touzi (1996): Option hedging and implicit volatilities. in stochastic volatilty models. Mathematical Finance 6, 279-302.

[22] CHRISTIAN BENDER, TOMMI SOTTINEN, AND ESKO VALKEILA ARBITRAGE WITH FRACTIONAL BROWNIAN MOTION?, Theory of Stochastic Processes Vol.12 (28), no.3-4, 2006

[23] Bernt Oksendal.Fractional Brownian Motion in Finance, Center of Mathematics for Applications (CMA) Department of Mathematics, University of Oslo, June 24, 2004

[24] Bernt Oksendal, An introduction to white noise theory and Malliavin Calculus for fractional brownian motion, Department of Mathematics, university of Bologna,2004

[25] M.B. Priestley, Spectral analysis and time series, vol. 1, Academic Press, 1981.

[26] Troels Roussau, The Bochner Minlos Theorem for nuclear spaces and an abstract white noise space,Johansen Louisiana State University, Spring 2003,

[27] Tommi Sottinen, Esko Valkeila (2003). On arbitrage and replication in the fractional BlackScholes pricing model. Statistics and Decisions: Vol. 21, Issue 2, pp. 93-108.

[28] David Siska, Stochastic Dierential Equations Driven by Fractional Brownian Motion - a White Noise., 1st September 2004

[29] PORTFOLIO OPTIMIZATION WITH CONSUMPTION IN A FRACTIONAL BLACK-SCHOLES MARKET, YALC IN SAROL, FREDERI G. VIENS, AND TAO ZHANG

[30] ITO FORMULAS FOR FRACTIONAL BROWNIAN MOTIONS,Robert J. Elliott

[31] FRACTIONAL WHITE NOISE CALCULUS AND APPLICATIONS TO FINANCE, Yaozhong hu and Bernt Oksendal,