NCUK
THE NORTHERN CONSORTIUM

WHBCWuhan IFYFM001

Semester 2 Examination

Further Mathematics : Vectors

Examination Session	**Time Allowed**
May 2013	1 hour

INSTRUCTIONS TO STUDENTS

- **Write your Student Number clearly on the Answer Booklet Provided**

1. **This exam is worth 5% of the overall marks for the course.**
2. **The time allowed for this exam is 1 hour.**
3. **This paper contains 6 questions.**
4. **Answer all questions.**
5. **The total number of marks for the exam is 50.**
6. **The marks for each question are indicated in square brackets.**
7. **Only approved calculators may be used.**
8. **No written material is allowed in the examination room.**
9. **No mobile phones are allowed in the examination room**

Vectors

Unit vector $\hat{\mathbf{a}}$ in the direction of $\mathbf{a}$

$\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$, where $|\mathbf{a}|$ is the modulus (magnitude) of $\mathbf{a}$.

$\vec{a}$ and $\overrightarrow{AB}$ are also used to denote vectors.

Scalar Product

$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

$\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,

then $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$, $\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = 0$,

$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$,

$\mathbf{a}.\mathbf{a} = |\mathbf{a}|^2 = a_1{}^2 + a_2{}^2 + a_3{}^2$.

If both $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors then $\mathbf{a}$ is perpendicular to $\mathbf{b}$ if $\mathbf{a}.\mathbf{b} = 0$.

Vectors

Vector Product

$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\mathbf{n}$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,

then $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$, $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$,

$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$.

If both $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors then $\mathbf{a}$ is parallel to $\mathbf{b}$ if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Moments as vectors

The moment about O of force $\mathbf{F}$ acting at position $\mathbf{r}$ is $\mathbf{r} \times \mathbf{F}$.

1) The lines $r=\begin{pmatrix}2\\-3\\1\end{pmatrix}+\lambda\begin{pmatrix}5\\1\\2\end{pmatrix}$ and $r=\begin{pmatrix}m\\2\\5\end{pmatrix}+\mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$ intersect.

a) Find the value of m. [3]

b) Find the point of intersection of the lines. [3]

c) Find the angle of intersection of the lines, giving your answer to the nearest 0.1^0. [3]

2) Given the line $r=i+3j+5k+\lambda(2i+4j+k)$ and the plane $r.(i-j+k)=0$, find:

a) the Cartesian form of the equation of the line, [2]

b) the Cartesian form of the equation of the plane, [2]

c) the angle of intersection of the line and the plane, and [3]

d) the point of intersection of the line and the plane. [5]

3) The planes $r.\begin{pmatrix}3\\2\\4\end{pmatrix}=13$ and $r.\begin{pmatrix}a\\1\\1\end{pmatrix}=4$ are perpendicular.

a) Find the value of a. [3]

b) Find the equation of the line of intersection of the two planes. [6]

4) Find the distance of the point $p=\begin{pmatrix}1\\5\\6\end{pmatrix}$ from the line $r=\begin{pmatrix}4\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\-3\end{pmatrix}$. [6]

5) The tetrahedron formed by the points with position vectors $A=\begin{pmatrix}2\\1\\3\end{pmatrix}$, $B=\begin{pmatrix}5\\2\\4\end{pmatrix}$, $C=\begin{pmatrix}3\\5\\1\end{pmatrix}$

and $D=\begin{pmatrix}6\\2\\k\end{pmatrix}$ has volume $V=1$. Find the integer value of k. [6]

6) Consider the triangle ABC. Let CA = **a** and CB = **b**.
Let D be the midpoint of CB and E the midpoint of CA.
Let G be the point of intersection of AD and BE.

a) Find AD in terms of **a** and **b**. [2]

b) Find BE in terms of **a** and **b**. [2]

Let CG = **a** + λAD = **b** + μBE.

c) Find the values of λ and μ and hence find the centroid of the triangle ABC. [4]

1) a)

$2+5\lambda = m+2\mu$ 1

$-3+\lambda = 2+\mu$ 2

$1+2\lambda = 5-\mu$ 3

2 + 3 $-2+3\lambda = 7$

$\lambda = 3$ 4

$\mu = -2$ 5

4&5 in 1 $2+15 = m-4$

$m = 21$. Method [2] Answer [1]

b)

$$r = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 17 \\ 0 \\ 7 \end{pmatrix}$$ Method [1] Answer [2]

or $$r = \begin{pmatrix} 21 \\ 2 \\ 5 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 17 \\ 0 \\ 7 \end{pmatrix}.$$

c)

$$\cos\theta = \frac{\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}.\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}}{\sqrt{5^2+1^2+2^2}\sqrt{2^2+1^2+1^2}} = \frac{9}{\sqrt{30}\sqrt{6}} = \frac{2}{2\sqrt{5}}$$

$\theta = 47.9^o$. Method [2] Answer [1]

2) a)

$$xi + yj + zk = i + 3j + 5k + \lambda(2i + 4j + k)$$

$$(x-1)i + (y-3)j + (z-5)k = \lambda(2i + 4j + k)$$

$$\lambda = \frac{(x-1)}{2} = \frac{(y-3)}{4} = (z-5)$$ Answer [2]

b)

$x - y + z = 0$ Answer [2]

c)

$$\sin\theta = \frac{\begin{pmatrix}2\\4\\1\end{pmatrix}.\begin{pmatrix}1\\-1\\1\end{pmatrix}}{\sqrt{2^2+4^2+1^2}\sqrt{1^2+1^2+1^2}} = \frac{-1}{3\sqrt{7}}$$

$\theta = 7.2^o$ Method [2] Answer [1]

d)

$$\left\{\begin{pmatrix}1\\3\\5\end{pmatrix} + \lambda\begin{pmatrix}2\\4\\1\end{pmatrix}\right\}.\begin{pmatrix}1\\-1\\1\end{pmatrix} = 0$$

$$(1+2\lambda)-(3+4\lambda)+(5+\lambda)=0$$

$$\lambda = 3$$

$$r = \begin{pmatrix}1\\3\\5\end{pmatrix} + 3\begin{pmatrix}2\\4\\1\end{pmatrix} = \begin{pmatrix}7\\15\\8\end{pmatrix}.$$

Method [3] Answer [2]

3) a)

$$\begin{pmatrix}3\\2\\4\end{pmatrix}.\begin{pmatrix}a\\1\\1\end{pmatrix} = 0$$

$a = -2$ Method [1] Answer [2]

b)

$$r = a + \lambda b$$

$$b = \begin{vmatrix} i & j & k \\ -2 & 1 & 1 \\ 3 & 2 & 4 \end{vmatrix} = i\begin{vmatrix}1 & 1\\2 & 4\end{vmatrix} - j\begin{vmatrix}-2 & 1\\3 & 4\end{vmatrix} + k\begin{vmatrix}-2 & 1\\3 & 2\end{vmatrix}$$

$b = 2i + 11j - 7k$ Method [1] Answer [2]

Let $z = 0$

$$\begin{pmatrix}3\\2\\4\end{pmatrix}.\begin{pmatrix}x\\y\\0\end{pmatrix} = 3x+2y=13$$

$$\begin{pmatrix}-2\\1\\2\end{pmatrix}.\begin{pmatrix}x\\y\\0\end{pmatrix} = -2x+y=4$$

$x = \frac{5}{7}$, $y = \frac{38}{7}$ or equivalent

$r = \frac{1}{7}(5i + 38j) + \lambda(2i + 11j - 7k)$ Method [2] Answer [1]

4)

$$(p - r).b = \begin{pmatrix} 1 - 4 - \lambda \\ 5 + 2\lambda \\ 6 - 1 + 3\lambda \end{pmatrix} . \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$$

$$= (-3 - \lambda) + (-10 - 4\lambda) + (-15 - 9\lambda)$$

$$= -28 - 14\lambda = 0$$
$$\lambda = -2$$

$$(p - r) = \begin{pmatrix} -3 + 2 \\ 5 - 4 \\ 5 - 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$|(p - r)| = \sqrt{1 + 1 + 1} = \sqrt{3}$ Method [4] Answer [2]

OR

$$d = \frac{|(p - a) \times b|}{|b|}$$

$$d = \frac{\left| \left\{ \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \right\} \times \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \right|}{\sqrt{1^2 + (-2)^2 + (-3)^2}} = \frac{\left| \begin{pmatrix} -3 \\ 5 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \right|}{\sqrt{1^2 + (-2)^2 + (-3)^2}}$$

$$\begin{pmatrix} -3 \\ 5 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = \begin{vmatrix} i & j & k \\ -3 & 5 & 5 \\ 1 & -2 & -3 \end{vmatrix} = i \begin{vmatrix} 5 & 5 \\ -2 & -3 \end{vmatrix} - j \begin{vmatrix} -3 & 5 \\ 1 & -3 \end{vmatrix} + k \begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix}$$

$$= -5i - 4j - k$$

$d = \frac{|-5i - 4j - k|}{\sqrt{1^2 + (-2)^2 + (-3)^2}} = \frac{\sqrt{42}}{\sqrt{14}} = \sqrt{3}$ Method [4] Answer [2]

5)

$$V = \frac{1}{6}|(D-A).(C-A)\times(B-A)|$$

$$D-A = \begin{pmatrix}6\\2\\k\end{pmatrix} - \begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}4\\1\\k-3\end{pmatrix}$$

$$C-A = \begin{pmatrix}3\\5\\1\end{pmatrix} - \begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}1\\4\\-2\end{pmatrix}$$

$$B-A = \begin{pmatrix}5\\2\\4\end{pmatrix} - \begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}3\\1\\1\end{pmatrix}$$

$$V = \frac{1}{6}\left|\begin{pmatrix}4\\1\\k-3\end{pmatrix}.\begin{pmatrix}1\\4\\-2\end{pmatrix}\times\begin{pmatrix}3\\1\\1\end{pmatrix}\right|$$

$$V = \frac{1}{6}\begin{vmatrix}4 & 1 & k-3\\1 & 4 & -2\\3 & 1 & 1\end{vmatrix} = \frac{1}{6}(50-11k) = 1$$

$k = 4$. Method [4] Answer [2]

6) a)

$AD = -a + b/2$ Answer [2]

b)

$BE = a/2 - b$ Answer [2]

c)

$$CG = a + \lambda AD = a + \lambda(-a + b/2) = a(1-\lambda) + \lambda b/2$$

$$CG = b + \mu BE = b + \mu(a/2 - b) = \mu a/2 + b(1-\mu)$$

coefficient of $a : (1-\lambda) = \mu/2$

coefficient of $b : \lambda/2 = (1-\mu)$

$\lambda = \mu = 2/3$

centroid at $(a+b)/3$. Method [2] Answer [2]