Section A Differential Eqns 2012

A 1

$$(x^2+4)\cos y \frac{dy}{dx} = 1$$

$$\int \cos y \, dy = \int \frac{1}{x^2+4} dx$$ [2]

[1]

$$\sin y = \frac{1}{2}\tan^{-1}(x/2) + C.$$

when $x = \pi/2$, $y = 0$

$$0 = \frac{1}{2}\tan^{-1}(\pi/4) + C = 1/2 + C \quad \therefore C = -1/2$$ [1]

$\underline{\sin y = 1/2 \tan^{-1}(x/2) - 1/2}$ in implicit form. [2]

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A 2

$$\frac{dy}{dx} + 4y = 8x + 6.$$

$m + 4 = 0$

$y_c = Ae^{-4x}.$ [1]

$y_p = Bx + C$ $\quad$ $4y_p = 4Bx + 4C$

$y_p' = B$ $\quad$ $y_p' = B$

$8x + 6.$

$B = 2, C = 1.$ [2]

$y = Ae^{-4x} + 2x + 1.$

$y' = -4Ae^{-4x} + 2$

$y'(0) = -4A + 2 = -6 \Rightarrow A = 2.$ [1]

$\underline{y = 2e^{-4x} + 2x + 1.}$ [2]

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A3 $x\frac{dy}{dx} + y = x\cos x$

$$\left[\ \frac{dy}{dx} + \frac{y}{x} = \cos x. \right.$$

integrating factor is $e^{\int \frac{1}{x}dx} = e^{\ln x} = x.$

$$\left. x\frac{dy}{dx} + y = x\cos x \ \right]$$

not necessary if spotted.

$$\frac{d}{dx}(xy) = x\cos x$$ [2]

$$xy = \int x\cos x\,dx$$

$$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}dx$$

$u = x$
$du = 1$
$dv = \cos x$
$v = \sin x$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x. + c.$$

[2]

$$xy = x\sin x + \cos x + c$$

$$y = \sin x + \frac{\cos x}{x} + \frac{c}{x}$$

[2]

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Section B

B1 $$6\frac{d^2y}{dx^2} - 13\frac{dy}{dx} + 6y = 0.$$

$$6m^2 - 13m + 6 = 0 \qquad [1]$$

$$m = \frac{13 \pm \sqrt{169 - 144}}{12} = \frac{13 \pm 5}{12} = 2/3 \text{ or } 3/2 \qquad [2]$$

$$\underline{y = Ae^{3/2x} + Be^{2/3x}} \qquad [2] \quad 5$$

B2 $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 26 = 0. \qquad [1]$$

$$m^2 + 2m + 26 = 0.$$

$$m = \frac{-2 \pm \sqrt{4 - 104}}{2} = \frac{-2 \pm \sqrt{-100}}{2} = -1 \pm 5i \qquad [2]$$

$$\underline{y = e^{-x}(A\cos 5x + B\sin 5x)} \qquad [2] \quad 5$$

B3 $$9\frac{d^2y}{dx^2} - 24\frac{dy}{dx} + 16y = 0.$$

$$9m^2 - 24m + 16 = 0 \qquad [1]$$

$$m = \frac{24 \pm \sqrt{24^2 - 4.9.16}}{18} = 4/3 \qquad [2]$$

$$\underline{y = (A + Bx)e^{4x/3}} \qquad [2] \quad 5$$

Section C

C1 $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 8y = 8e^{4x}$.

$m^2 - 6m + 8 = 0$

$(m-4)(m-2) = 0$

$y_c = Ae^{2x} + Be^{4x}$. [3]

$y_p = Cxe^{4x}$

$y_p' = 4Cxe^{4x} + Ce^{4x}$

$y_p'' = 16Cxe^{4x} + 4Ce^{4x} + 4Ce^{4x}$

$y_p'' - 6y_p' + 8y_p = 8Ce^{4x} - 6Ce^{4x} = 2Ce^{4x} = 8e^{4x}$ $\therefore C = 4$. [3]

$y = (A + 4x)e^{4x} + Be^{2x}$ [2]

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C3. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = -6x^2 + 14x - 18$

$m^2 + m - 6 = 0$

$(m+3)(m-2) = 0$

$y_c = Ae^{2x} + Be^{-3x}$. [3]

$y_p = Cx^2 + Dx + E$

$y_p' = 2Cx + D$

$y_p'' = 2C$

$$
\begin{array}{rrrr}
-6y_p = & -6Cx^2 & -6Dx & -6E \\
y_p' = & & 2Cx & +D \\
y_p'' = & & & 2C \\
\hline
 & -6x^2 & +14x & -18
\end{array}
$$

[3]

$C = 1,\ D = -2,\ E = 3.$

$y = Ae^{2x} + Be^{-3x} + x^2 - 2x + 3$ [2]

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Section C

C2

$$\frac{d^2y}{dx^2} + 4y = 12\cos 2x$$

$m^2 + 4 = 0 \quad m = \pm 2i$

$y_c = A\sin 2x + B\cos 2x.$ [3]

$y_p = Cx\sin 2x + Dx\cos 2x$

$y_p' = C\sin 2x + D\cos 2x + 2Cx\cos 2x - 2Dx\sin 2x$

$y_p'' = 2C\cos 2x - 2D\sin 2x + 2C\cos 2x - 2D\sin 2x - 4Cx\sin 2x - 4Dx\cos 2x.$ [3]

$4y_p + y_p'' = 4C\cos 2x - 4D\sin 2x = 12\cos 2x \qquad D = 0,\ C = 3.$

$y = A\sin 2x + (B + 3x)\cos 2x$ [2] 8

Section D

D1

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = -24x + 10.$$

$m^2 - 2m - 8 = 0$

$(m+2)(m-4) = 0$

$y_c = Ae^{-2x} + Be^{4x}$ [3]

$y_p = Cx + D \qquad -8y_p = -8Cx - 8D$

$y_p' = C \qquad -2y_p' = -2C$

$\qquad\qquad\qquad -24x + 10$ [3]

$C = 3,\ D = -2.$

$y = Ae^{-2x} + Be^{4x} + 3x - 2.$

$y' = -2Ae^{-2x} + 4Be^{4x} + 3.$

~~y tends~~ y' remains finite for large $x \Rightarrow B = 0.$ [4]

$y' = -2Ae^{-2x} + 3.$

$y(0) = -2A + 3 = 15 \Rightarrow A = -6.$ 12

$y = -6e^{-2x} + 3x - 2$ [2]

Section D

D2

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 5\sin 3x - 25\cos 3x$$

$m^2 + 3m + 2 = 0$

$(m+2)(m+1) = 0$

$y_c = Ae^{-x} + Be^{-2x}$ [3]

$y_p = C\sin 3x + D\cos 3x$

$y_p' = -3D\sin 3x + 3C\cos 3x$

$y_p'' = -9C\sin 3x - 9D\cos 3x$

$2y_p = 2C\sin 3x + 2D\cos 3x$

$3y_p' = -9D\sin 3x + 9C\cos 3x$

$y_p'' = -9C\sin 3x - 9D\cos 3x$

$+ \quad (-7C - 9D)\sin 3x + (9C - 7D)\cos 3x$

$= 5\sin 3x - 25\cos 3x$ [3]

$\left.\begin{aligned} -7C - 9D &= 5 \\ 9C - 7D &= -25 \end{aligned}\right\} C = -2,\ D = 1.$

$y_p = -2\sin 3x + \cos 3x.$

$y = Ae^{-x} + Be^{-2x} - 2\sin 3x + \cos 3x.$

$y(0) = \underline{A + B + 1 = 4}$

$y' = -Ae^{-x} - 2Be^{-2x} - 6\cos 3x - 3\sin 3x$

$y'(0) = \underline{-A - 2B - 6 = 0}.$ [4]

$B = -9,\ A = 12.$

$\underline{y = 12e^{-x} - 9e^{-2x} - 2\sin 3x + \cos 3x}$ [2]

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