Vectors: May 2012

1) Cartesian equation of the plane.

$\underset{\sim}{r} \cdot (4\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}) = 6$

$(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}) \cdot (4\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}) = 6$

$\underline{4x - y + 3z = 6}$ [3]

2) Cartesian equation of the line.

$\underset{\sim}{r} = 3\underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k} + \lambda(\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k})$

$(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}) - (3\underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k}) = \lambda(\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k})$ } [1]

$\underline{x - 3 = \frac{y-1}{-2} = \frac{z+4}{-1}}$ [2]

3) Distance of the point from the plane.

$\text{distance} = \underset{\sim}{p} \cdot \hat{u} - d$

$= \frac{\underset{\sim}{p} \cdot \underset{\sim}{n} - D}{|n|}$

$= \frac{(2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}) \cdot (4\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}) - 6}{\sqrt{4^2 + 1^2 + 3^2}}$ } [3]

$= \frac{8 + 3 + 3 - 6}{\sqrt{26}} = \frac{8}{\sqrt{26}} = \underline{\frac{4\sqrt{26}}{13}}$ [2]

or equivalent.

4) point of intersection of line and plane.

$l: \underset{\sim}{r} = (3\underset{\sim}{i} + j - 4\underset{\sim}{k}) + \lambda(\underset{\sim}{i} - 2j - \underset{\sim}{k})$

$\Pi: \underset{\sim}{r} \cdot (4\underset{\sim}{i} - j + 3\underset{\sim}{k}) = 6.$

l in Π. $[(3+\lambda)\underset{\sim}{i} + (1-2\lambda)j + (-4-\lambda)\underset{\sim}{k}] \cdot (4\underset{\sim}{i} - j + 3\underset{\sim}{k}) = 6.$

$(3+\lambda)4 + (1-2\lambda)(-1) + (-4-\lambda)(3) = 6.$ [4]

$\vdots$

$\lambda = 7/3.$

λ in l $\quad (3\underset{\sim}{i} + j - 4\underset{\sim}{k}) + 7/3(\underset{\sim}{i} - 2j - k)$

$= \frac{1}{3}(16\underset{\sim}{i} - 11j - 19\underset{\sim}{k})$ [2].

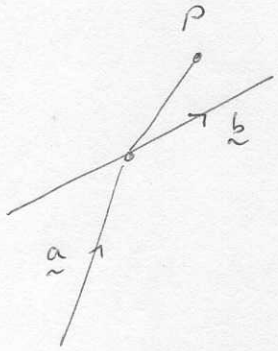
5) angle of intersection of line and plane.

$\sin\theta = \frac{\underset{\sim}{n} \cdot \underset{\sim}{b}}{|n||b|} = \frac{(4\underset{\sim}{i} - j + 3\underset{\sim}{k}) \cdot (\underset{\sim}{i} - 2j - \underset{\sim}{k})}{\sqrt{26} \quad \sqrt{6}.}$ [2]

$= \frac{4+2-3}{\sqrt{26}\sqrt{6}} = \dots = 0.2419\dots$

$\sin^{-1}(0.2419) = 13.9°$ (3 s.f.) [1].

6) equation of the plane containing the point and the line.



$$p - a = (2i - 3j + k) - (3i + j - 4k)$$

$$= -i - j + 5k$$ lies ine the plane. [1]

b lies in the plane.

$$n = (p - a) \times b$$

$$= \begin{vmatrix} i & j & k \\ -1 & -4 & 5 \\ 1 & -2 & -1 \end{vmatrix} = i \begin{vmatrix} -4 & 5 \\ -2 & -1 \end{vmatrix} - j \begin{vmatrix} -1 & 5 \\ 1 & -1 \end{vmatrix} + k \begin{vmatrix} -1 & -4 \\ 1 & -2 \end{vmatrix}$$

$$= 14i + 4j + 6k \;\; // \;\; 7i + 2j + 3k$$ [3]

$$\underline{n = 7i + 2j + 3k}$$

a is a point in the plane

$$a \cdot n = (3i + j - 4k) \cdot (7i + 2j + 3k)$$

$$= 21 + 2 - 12 = 11.$$ [2]

$$\underline{r \cdot (7i + 2j + 3k) = 11}$$ [2]

7) distance of the point from the line.

$(\underset{\sim}{p} - \underset{\sim}{r}) \cdot \underset{\sim}{b} = 0.$

$$\left[(2\underline{i} - 3\underline{j} + \underline{k}) - \left((3+\lambda)\underline{i} + (1-2\lambda)\underline{j} + (-4-\lambda)\underline{k}\right)\right] \cdot (\underline{i} - 2\underline{j} - \underline{k}) = 0.$$

$$\left((-1-\lambda)\underline{i} + (-4+2\lambda)\underline{j} + (5+\lambda)\underline{k}\right) \cdot (\underline{i} - 2\underline{j} + \underline{k}) = 0$$

[3]

$$\vdots$$

$$\lambda = 1/3.$$

$$\underset{\sim}{p} - \underset{\sim}{r} = (-1 - 1/3)\underline{i} + (-4 + 2/3)\underline{j} + (5 + 1/3)\underline{k}$$

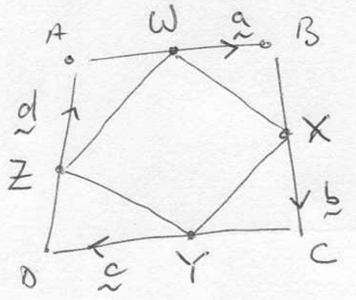
$$= 1/3(-4\underline{i} - 10\underline{j} + 16\underline{k}).$$

$$= 2/3(-2\underline{i} - 5\underline{j} + 8\underline{k}).$$ [2]

$$|\underset{\sim}{p} - \underset{\sim}{r}| = 3/3\sqrt{2^2 + 5^2 + 8^2}$$

$$= 2/3\sqrt{93}$$ [2].

8) Varignon's theorem.



i) $\underline{a} + \underline{b} + \underline{c} + \underline{d} = \underline{0}.$ [1]

ii) $WX = \frac{1}{2}(\underline{a} + \underline{b})$ [1]

$ZY = -\frac{1}{2}(\underline{d} + \underline{c}) = \frac{1}{2}(\underline{a} + \underline{b})$ [1]

(iii) $XY = \frac{1}{2}(\underline{b} + \underline{c})$ [1]

$WZ = -\frac{1}{2}(\underline{a} + \underline{d}) = \frac{1}{2}(\underline{b} + \underline{c})$ [1]

(iv) WXYZ is a parallelogram. [1]

9)

~~BA = (i 2j~~

i) $B - A = (\underline{i} - 2\underline{j} + \underline{k}) - (2\underline{i} + \underline{j} + 3\underline{k}) = (-\underline{i} - 3\underline{j} - 2\underline{k})$

$C - A = (4\underline{i} - 3\underline{j} - \underline{k}) - (2\underline{i} + \underline{j} + 3\underline{k}) = (2\underline{i} - 4\underline{j} - 4\underline{k})$

$$(B - A) \times (C - A) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -3 & -2 \\ 2 & -4 & -4 \end{vmatrix}$$

$$= \underline{i} \begin{vmatrix} -3 & -2 \\ -4 & -4 \end{vmatrix} - \underline{j} \begin{vmatrix} -1 & -2 \\ 2 & -4 \end{vmatrix} + \underline{k} \begin{vmatrix} -1 & -3 \\ 2 & -4 \end{vmatrix}$$

$$= 4\underline{i} - 8\underline{j} + 10\underline{k}$$ [3]

area of triangle $= \frac{1}{2} |(B - A) \times (C - A)|$

$$= \frac{1}{2} \sqrt{4^2 + 8^2 + 10^2}$$

$$= \frac{1}{2} \sqrt{180} = \underline{3\sqrt{5}}$$ [2]

ii) $D - A = (-2\underline{i} + 3\underline{j} - 2\underline{k}) - (2\underline{i} + \underline{j} + 3\underline{k}) = (-4\underline{i} + 2\underline{j} - 5\underline{k})$

$(D - A) . (B - A) \times (C - A)$

$$= (-4\underline{i} + 2\underline{j} - 5\underline{k}) . (4\underline{i} - 8\underline{j} + 10\underline{k})$$

$$= -16 - 16 - 50 = -82.$$ [2]

volume of tetrahedron $= 1/6 \, |(D - A) . (B - A) \times (C - A)|$

$$= 82/6 = \underline{41/3}$$ [2].