Vectors

Unit vector $\hat{\mathbf{a}}$ in the direction of $\mathbf{a}$

$\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$, where $|\mathbf{a}|$ is the modulus (magnitude) of $\mathbf{a}$.

$\vec{a}$ and $\overrightarrow{AB}$ are also used to denote vectors.

Scalar Product

$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

$\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,
then $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$, $\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i} = 0$,
$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$,
$\mathbf{a}.\mathbf{a} = |\mathbf{a}|^2 = a_1{}^2 + a_2{}^2 + a_3{}^2$.
If both $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors then $\mathbf{a}$ is perpendicular to $\mathbf{b}$ if $\mathbf{a}.\mathbf{b} = 0$.

Vectors

Vector Product

$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\mathbf{n}$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,
then $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$, $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$,
$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$.

If both $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors then $\mathbf{a}$ is parallel to $\mathbf{b}$ if
$\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Moments as vectors

The moment about O of force $\mathbf{F}$ acting at position $\mathbf{r}$ is $\mathbf{r} \times \mathbf{F}$.

A point is given by the position vector $\mathbf{p} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$.

A line is given by the vector equation l: $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$.

A plane is given by the vector equation Π_1: $\mathbf{r}.(\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 3$.

Find:

1) the cartesian equation of the line, l. [3]

2) the cartesian equation of the plane, Π_1. [3]

3) the angle of intersection of the line, l and the plane, Π_1. Give your answer in degrees to 3 significant figures. [3]

4) the point of intersection of the line, l and the plane, Π_1. [6]

5) the distance of the plane, Π_1 from the origin. Give your answer in exact form. [4]

6) the distance of the point, p from the plane, Π_1. Give your answer in exact form. [5]

7) the distance of the point, p from the line, l. Give your answer in exact form. [7]

8) the equation of the plane Π_2 containing the point, p and the line, l. Give your answer in the form $\mathbf{r}.\mathbf{n} = D$. [8]

9) the angle between the planes Π_1 and Π_2. [3]

10) the line of intersection of the planes Π_1 and Π_2. Give your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. [8]

1 $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

$\mathbf{x} + \mathbf{y} + \mathbf{z} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

$$\frac{x-2}{1}=\frac{y-1}{-3}=\frac{z+3}{3}$$

or $\frac{x-2}{1}=\frac{1-y}{3}=\frac{z+3}{3}$.

2 $\mathbf{r}.(\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 3$

$(\mathbf{x} + \mathbf{y} + \mathbf{z}).(\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 3$

$\mathbf{x} - y - 2z = 3$.

3 $\sin\theta = \frac{n.b}{|n||b|} = \frac{(i-j-2k)(i-3j+3k)}{\sqrt{1+1+4}\sqrt{1+9+9}}$

$\sin\theta = \frac{1+3-6}{\sqrt{6}\sqrt{19}} = \frac{-2}{\sqrt{6}\sqrt{19}} = -0.1873$

$\theta = \sin^{-1}(-0.1873) = -10.8^{o}$

Accept $+\ 10.8^{o}$ or $-\ 10.8^{o}$.

4 $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

$\mathbf{r}.(\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 3$

$(2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})).(\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 3$

$(2 + \lambda)(1) + (1 - 3\lambda)(-1) + (-3 + 3\lambda)(-2) = 3$

$2 + \lambda - 1 + 3\lambda + 6 - 6\lambda = 3$

$7 - 2\lambda = 3$

$\lambda = 2$

$\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

$\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + 2(\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

$\mathbf{r} = 4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$.

5 $\frac{r.n}{|n|} = d = \frac{3}{\sqrt{1+1+4}} = \frac{3}{\sqrt{6}} = \frac{\sqrt{6}}{2}$ or equivalent.

6 distance is $\frac{p.n}{|n|} - d$.

$\frac{p.n}{|n|} - d = \frac{(3i - j + k)(i - j - 2k)}{\sqrt{6}} - \frac{3}{\sqrt{6}}$

$= \frac{(3+1-2)-3}{\sqrt{6}} = \frac{-1}{\sqrt{6}} = \frac{-\sqrt{6}}{6}$ .Accept +ve or –ve value or surd equivalent.

7 **p** = 3**i** **-** **j** + **k**

r = 2**i** + **j** **-** 3**k** + λ(**i** **-** 3**j** + 3**k**) = **a** + λ**b**.

(p – r).b = 0.

[(3**i** **-** **j** + **k)** – (2**i** + **j** **-** 3**k** + λ(**i** **-** 3**j** + 3**k**))]. (**i** **-** 3**j** + 3**k**) = 0.

(3 – 2 - λ)(1) + (–1 – 1 + 3λ)(–3) + (1 + 3 – 3λ)(3) = 0

(1 – λ) + (6 – 9λ) + (12 – 9λ) = 0

19 – 19λ = 0

λ = 1

(p – r) = (3**i** **-** **j** + **k)** – (2**i** + **j** **-** 3**k** + λ(**i** **-** 3**j** + 3**k**))

= (3**i** **-** **j** + **k)** – (2**i** + **j** **-** 3**k** + (**i** **-** 3**j** + 3**k**))

= j + k

|(p – r)| = $\sqrt{2}$.

8 n = **(p – a)xb**

n = ((3**i** **-** **j** + **k) – (**2**i** + **j** **-** 3**k))x(** **i** **-** 3**j** + 3**k**)

n = (i – 2j + 4k)x(i – 3j + 3k)

n = $\begin{vmatrix} i & j & k \\ 1 & -2 & 4 \\ 1 & -3 & 3 \end{vmatrix} = i\begin{vmatrix} -2 & 4 \\ -3 & 3 \end{vmatrix} - j\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} + k\begin{vmatrix} 1 & -2 \\ 1 & -3 \end{vmatrix}$

n = $= 6i + j - k$

or use the formula provided

r.n = D

choose a (3**i** - **j** + **k**).(**6i + j – k**) = 18 – 1 – 1 = 16

or choose p (2**i** + **j** - 3). **(6i + j – k) =** 12 + 1 + 3 = 16

so **r**. **(6i + j – k) =** 16.

9 $\cos\theta = \dfrac{n_1.n_2}{|n_1||n_2|} = \dfrac{(i - j - 2k)(6i + j - k)}{\sqrt{6}\sqrt{38}}$

$= \dfrac{6 - 1 + 2}{\sqrt{6}\sqrt{38}} = \dfrac{7}{\sqrt{6}\sqrt{38}} = 0.4636$

$\theta = \cos^{-1}(0.4636) = 62.6^o$.

10 **b** = $\mathbf{n}_1$x$\mathbf{n}_2$ = $\begin{vmatrix} i & j & k \\ 1 & -1 & -2 \\ 6 & 1 & -1 \end{vmatrix} = i\begin{vmatrix} -1 & -2 \\ 1 & -1 \end{vmatrix} - j\begin{vmatrix} 1 & -2 \\ 6 & -1 \end{vmatrix} + k\begin{vmatrix} 1 & -1 \\ 6 & 1 \end{vmatrix}$

b = $3i - 11j + 7k$

or use the formula provided

to find a, let x = 0 or y = 0 or z = 0.

If x = 0, **a** = (y**j** + z**k**), and **a**.$\mathbf{n}_1$ = D_1 and **a**.$\mathbf{n}_2$ = D_2.

(y**j** + z**k**). (**i** - **j** - 2**k**) = 3 - y – 2z = 3 (1)

(y**j** + z**k**). **(6i + j – k) =** 16 y – z = 16 (2)

(1) & (2) give y = 29/3 & z = -19/3

r = **a** + λ**b** = (1/3)(29**j** – 16**k**) + λ(3**i** – 11**j** + 7**k**).